Advances in Monte Carlo for Stochastic Networks

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Outline

Goal of the Program

Goal: **Enable the design & analysis of good Monte Carlo methods for stochastic networks.**

- Design & analysis
- Good Monte Carlo
- Stochastic networks

Why Monte Carlo?

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$$\widehat{\alpha}_n(d) := \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{Y}^{(i)}) \longrightarrow \alpha_n(d)$$

at rate $Var(f(\mathbf{Y}))^{1/2} / n^{1/2}$.

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- Cost to estimate $\alpha(d)$ with ε-precision

$$Var(f(\mathbf{Y})) \times \text{Cost of evaluating}(f) \times O(1/\varepsilon^2)$$

What is Good? Creutzig et al (2007)

- $\alpha(d) := Ef(Y_1, ..., Y_d) \in (-\infty, \infty)$
- Cost to estimate $\alpha(d)$ with ε-precision

Good Monte Carlo =

$$\widetilde{O}(1/\varepsilon^2) \times Var(f(\mathbf{Y})) \times \text{(function specific cost) function evaluations}$$

Stochastic Networks

- Examples in heavy-traffic modeling:

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2. Generalized Halfin-Whitt diffusions
3. Josh's stochastic convolution equation...
4. Many server queues

Outline

Challenge: Continuous Stochastic Networks

- Fully continuous random object
  $\mathbf{Y} = ((Y_1(t), Y_2(t), ..., Y_d(t)) : 0 \leq t \leq T)$.
- Density of $\mathbf{Y}(t) = (Y_1(t), ..., Y_d(t))$ unknown.
- $f(\mathbf{Y})$ fully path dependent.

Yet... still want to estimate $\alpha = Ef(\mathbf{Y})$ using only $\widetilde{O}(1/\varepsilon^2)$ simple random numbers

Brief Literature Review

WARNING: References missing, text cited instead.

	Reference	Process	Functions – f(.)	Error rate
	Kloeden & Platten (1992)	Euler - Diffusions no reflection	Marginal & smooth	**$O(1/\varepsilon^3)$**
	Asmussen, Glynn, Pitman (1995)	1d Diffusions & reflection	Marginal & smooth	**$O(1/\varepsilon^4)$, $O(1/\varepsilon^3)$**
	Giles (2008)	Diffusions no reflection	Marginal & smooth	**$\tilde{O}(1/\varepsilon^2)$**
	Beskos & Roberts (1995)	1d Diffusions with jumps & no refl.	Marginal & smooth	**$O(1/\varepsilon^2)$**

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- **Skorokhod problem:** Solve for $(\mathbf{Y}(\cdot), \mathbf{L}(\cdot))$ satisfying

$$\begin{aligned} d\mathbf{Y}(t) &= d\mathbf{X}(t) + Rd\mathbf{L}(t), \ \mathbf{Y}(0) = y_0 \\ \mathbf{Y}(t) &\geq 0 \text{ componentwise} \\ dL_j(t) &\geq 0 \text{ non-decreasing for } j = 1, ..., d \\ Y_j(t)\, dL_j(t) &= 0 \text{ –> } (\text{"}dL_j(t) = \delta_{\{0\}}(Y_j(t))\, dt\text{"}) \end{aligned}$$

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- Call $\Phi_T(\mathbf{X}) = \mathbf{Y}_T = (\mathbf{Y}(t) : 0 \leq t \leq T)$
- $\Phi_T(\cdot)$ = Lipschitz continuous (uniform metric), constant K_Φ.

An Illustrative Example

Say one picks $f(\mathbf{Y}) = \sum_{j=1}^{d} \max_{0 \le t \le 1} Y_j^2(t)$

We will illustrate good Monte Carlo (i.e. $\widetilde{O}\left(1/\varepsilon^2\right)$ *rate) to estimate* $\alpha = Ef(\mathbf{Y})$

A Two Dimensional "True" Reflected Brownian Path

- $\mathbf{d}(\mathbf{Y}, \mathbf{Y}_\varepsilon) \leq \varepsilon = 10^{-3}$ *with probability one,*
- $\mathbf{d}(\cdot)$ uniform metric
- Producing $\mathbf{Y}_\varepsilon$ takes $\widetilde{O}\left(1/\varepsilon^2\right)$ standard Normal random variables (WILL SEE HOW)

Applying Multi-level Monte Carlo

- f locally Lipschitz continuous: If $\mathbf{d}(\omega, \omega') \leq 1$

$$\left| f(\omega) - f\left(\omega'\right) \right| \leq \underbrace{3d \times (2\,||\omega||_{\infty} + 1)}_{K_f(\omega)} \times \mathbf{d}(\omega, \omega')$$

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- Note that $\mathbf{d}\left(\mathbf{Y}, \mathbf{Y}_{\varepsilon(L)}\right) = O\left(2^{-L}\right)$ and $\mathbf{d}\left(\mathbf{Y}_{\varepsilon(n)}, \mathbf{Y}_{\varepsilon(n-1)}\right) = O\left(2^{-n}\right)$

Combining TES & Multi-level Monte Carlo

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$$\sum_{n=1}^{L} \frac{1}{N(n)} \sum_{i=1}^{N(n)} R^{(i)}(n)$$

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$$\begin{aligned} |Bias| &= |ER(L)| \leq EK_f(\mathbf{Y}) \times \varepsilon(L) = 3d \times E(||\mathbf{Y}||_\infty) \times 2^{-L} \\ Var(R(n)) &\leq 18d^2 \times K_\Phi \times E((2||\mathbf{Y}||_\infty + 1)^2) \times 2^{-2n} \end{aligned}$$

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- Put $N(n) = C2^{2(L-n)}$:

$$\text{Error}^2 = O(2^{-2L} + \sum_{n=1}^{L} \frac{2^{-2n}}{C2^{2L-2n}}) = O\left(L2^{-2L}\right) = \widetilde{O}\left(2^{2L}\right)$$

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- *Cost analysis:*
  $O(\sum_{n=1}^{L} N(n) \times \kappa_{TES}(2^{-n})) = \widetilde{O}(\sum_{n=1}^{L} 2^{2L-2n} \times 2^{2n}) = \widetilde{O}\left(2^{2L}\right)$

Summary: Result

Theorem: Suppose that

1. *Given a TES family* $\{\mathbf{X}_\varepsilon\}_{\varepsilon>0}$ *with cost* $\kappa_{TES}(\varepsilon) = \widetilde{O}(\varepsilon^{-2})$ *with respect to metric* $d(\cdot)$
2. $f(\cdot)$ satisfies $|f(\omega) - f(\omega')| \leq K_f(\omega) \times \mathbf{d}(\omega, \omega')$ & $EK(\mathbf{X})^2 < \infty$.
3. Cost of evaluating $f(\mathbf{X}_\varepsilon)$ is $O(\kappa_{TES}(\varepsilon))$.

Then, we can estimate $Ef(\mathbf{X})$ with ε at a cost $\widetilde{O}(\varepsilon^{-2})$.

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- Assume $\sum_{k=1}^{\infty} E^{1/2}\left|f(\mathbf{Y}_k) - f(\mathbf{Y}_{k-1})\right|^2 < \infty$

$$Ef(\mathbf{Y}) - Ef(\mathbf{Y}_0) = \sum_{k=1}^{\infty} E(f(\mathbf{Y}_k) - f(\mathbf{Y}_{k-1}))$$

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$$Ef(\mathbf{Y}) - Ef(\mathbf{Y}_0) = \sum_{k=1}^{\infty} E(f(\mathbf{Y}_k) - f(\mathbf{Y}_{k-1}))$$

- Simplicity $Ef(\mathbf{Y}_0) = 0$ & let $Z_k^{(m)} := (f(\mathbf{Y}_k^{(m)}) - f(\mathbf{Y}_{k-1}^{(m)}))$ all independent

$$W = \sum_{k=1}^{\infty} \frac{1}{n(k)} \sum_{m=1}^{n(k)} (f(\mathbf{Y}_k^{(m)}) - f(\mathbf{Y}_{k-1}^{(m)})) = \sum_{k=1}^{\infty} \frac{1}{n(k)} \sum_{m=1}^{n(k)} Z_k^{(m)}$$

Multilevel Monte Carlo

- Let $\mathcal{F}_n = \sigma\{Z_k^{(m)} : k \leq n, m \leq n(k)\}$

$$E(W|\mathcal{F}_n) = Ef(\mathbf{Y}) - Ef(\mathbf{Y}_n) + \sum_{k=1}^{n} \frac{1}{n(k)} \sum_{m=1}^{n(k)} Z_k^{(m)}$$

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- Given ε, set $L = L(\varepsilon)$ such that $|Ef(\mathbf{Y}) - Ef(\mathbf{Y}_L)| \leq \varepsilon$

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- Cost to produce Z_k, say $h(k)$ & $Var(Z_k) = \sigma(k)^2$,

$$\min \sum_{k=1}^{L} n(k)\, h(k)$$

$$\sum_{k=1}^{L} \frac{1}{n(k)} \sigma(k)^2 \leq 4 \times \varepsilon^2$$

Multilevel Monte Carlo

- Optimal allocation

$$n(k) = \frac{\sigma(k)}{h(k)^{1/2}} \times \sum_{k=1}^{L} \sigma(k)\, h(k)^{1/2} \times \frac{1}{4\varepsilon^2}$$

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- Computational cost

$$\begin{aligned} \sum_{k=1}^{L} n(k)\, h(k) &= \frac{1}{4\varepsilon^2} \times \left(\sum_{k=1}^{L(\varepsilon)} \sigma(k)\, h(k)^{1/2} \right)^2 \\ \left| Ef(\mathbf{Y}) - Ef\left(\mathbf{Y}_{L(\varepsilon)}\right) \right| &\leq \varepsilon. \end{aligned}$$

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- In particular, $O\left(\varepsilon^{-2}\right)$ complexity if

$$\sum_{k=1}^{\infty} \sigma(k)\, h(k)^{1/2} < \infty.$$

Outline

High Level Idea: Tolerance Enforced Couplings

- $\mathbf{Y}$ <– difficult process of interest
- $\mathbf{Y} = \Phi(\mathbf{X})$, where $\Phi(\cdot)$ Lipschitz relative to metric $\mathbf{d}(\cdot)$
- $\mathbf{X}$ more tractable process.
- Construct couplings $(\{\mathbf{X}_{\varepsilon}\}_{\varepsilon>0}, \mathbf{X})$ such that

$$\mathbf{d}(\mathbf{X}, \mathbf{X}_{\varepsilon}) \leq \varepsilon$$

with probability one.

Bias Deletion...

- Say $f(\cdot)$ positive and locally Holder

$$\left|f(\omega) - f\left(\omega'\right)\right| \leq K_{f,\beta}(\omega)\,\mathbf{d}\left(\omega, \omega'\right)^{\beta}$$

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- Suppose T is any r.v. independent of everything & has density $g(\cdot)$

$$Ef(\mathbf{Y}) = E\int_0^{f(\mathbf{Y})} dt = E\int_0^{f(\mathbf{Y})} \frac{g(t)}{g(t)}dt = E\left(\frac{I(f(\mathbf{Y})>T)}{g(T)}\right).$$

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- So, simulate T & output

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- So, simulate T & output

$$\frac{I(f(\mathbf{Y})>T)}{g(T)}$$

- This can be simulated exactly: Keep decreasing ε until

$$\begin{aligned} f(\mathbf{Y}_\varepsilon) &> T+2\varepsilon^{\beta}K_{f,\beta}(\mathbf{Y}_\varepsilon) \text{ then } I(f(\mathbf{Y})>T)=1\\ f(\mathbf{Y}_\varepsilon) &< T-2\varepsilon^{\beta}K_{f,\beta}(\mathbf{Y}_\varepsilon) \text{ then } I(f(\mathbf{Y})>T)=0\end{aligned}$$

TES for Brownian Motion

- Wavelets construction for $(B(t) : 0 \leq t \leq 1)$

$$B(t) = \sum_{n=0}^{N} \lambda_n \Delta_n(t) Z_n + \sum_{n=N+1}^{\infty} \lambda_n \Delta_n(t) Z_n$$

where Z_n's are i.i.d. $N(0,1)$ and $\lambda_n = 2^{-j}/2$ assuming $n = 2^j + k$

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- Pick $r(n) = 4\log(n+e)^{1/2}$ & $P[|Z_n| > r(n) \text{ i.o.}] = 0$

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- Pick $r(n) = 4\log(n+e)^{1/2}$ & $P[|Z_n| > r(n) \text{ i.o.}] = 0$
- Simulate Z_n's jointly with the times $T_m = \min\{n > T_{m-1} : |Z_n| > r(n)\}$

Complexity of TES for Brownian Motion

- Get $N = \max\{T_k : T_k < \infty\}$ (again easy to simulate) & if $M = 2^J \geq N + 1/\varepsilon^2$

$$\begin{aligned}\sum_{n=M}^{\infty} \lambda_n \Delta_n(t)\,|Z_n| &\leq \sum_{j=J}^{\infty} 4\,(j+1)^{1/2} \log(2) \lambda_{2^j} \sum_{k=0}^{2^j-1} \Delta_{2^j+k}(t) \\ &= \sum_{j=J}^{\infty} 2\,(j+1)^{1/2} \log(2) 2^{-j/2} \\ &= \widetilde{O}\left(2^{-J/2}\right) = \widetilde{O}(\varepsilon).\end{aligned}$$

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$$\begin{aligned}
\sum_{n=M}^{\infty} \lambda_n \Delta_n(t) |Z_n| &\leq \sum_{j=J}^{\infty} 4(j+1)^{1/2} \log(2) \lambda_{2^j} \sum_{k=0}^{2^j-1} \Delta_{2^j+k}(t) \\
&= \sum_{j=J}^{\infty} 2(j+1)^{1/2} \log(2) 2^{-j/2} \\
&= \widetilde{O}\left(2^{-J/2}\right) = \widetilde{O}(\varepsilon).
\end{aligned}$$

- Indeed, $\kappa_{TES}(\varepsilon) = \widetilde{O}(\varepsilon^{-2})$ as claimed.

TES for Brownian Motion

Algorithm 1.1: Output M *(truncation)* **jointly with** Z_n's **given** ε

Step 0: Set, ε, $\rho = 4$, $G = 2\left\lceil \varepsilon^{-2} \right\rceil$, $S = []$.

Step 1: Set $U = 1$, $D = 0$. Simulate $V \sim U(0,1)$.

Step 2: While $U > V > D$, set $G \leftarrow G + 1$, $U \leftarrow P(|Z| \leq \rho\sqrt{\log G}) * U$ and $D \leftarrow (1 - G^{1-\rho^2/2})U$.

Step 3: If $V \geq U$, $S = [S, G]$ and return to Step 1.

Step 4: If $V \leq D$, $M = G$, $S = [S, G]$.

Step 5: If and only if $n \in S$, Z_n has law Z given $|Z| > \rho\sqrt{\log n}$.

Outline

Steady-state Analysis

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- IMPORTANT: For proof of Corollary one needs K_Φ independent of ω & time.

Steady-state Analysis: An Alternative Procedure

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- Rates of convergence?

Steady-state Simulation: Some Numerical Experiments

Estimated Mean in an 8 Dimensional Symmetric Network					
Correlation in B.M.	0	0.2	0.9	-0.05	-0.1
Estimation	**0.179**	**0.241**	**0.47**	**0.167**	**0.146**
Standard deviation	0.044	0.061	0.042	0.043	0.04
True value	**0.182**	**0.245**	**0.468**	**0.166**	**0.15**

Total # of 1 dimensional Gaussian r.v.'s simulated = 1.1 * 10^7

Steady-state Simulation: Some Numerical Experiments

- Dai, Nguyen, Reiman '94

Estimated Mean in a 10 Dimensional Network in Series

Station	1	2	3	4	5	6	7	8	9	10
Traffic int	**0.9**	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	**0.9**
Prelimit	**30**	3.2	2	1.7	1.5	1.5	1.4	1.3	1.3	**16**
QNET	**32**	3.2	1.4	1.1	1	1	0.9	0.9	0.9	**8.1**
Ours	**32**	2.7	1.2	1	0.9	0.8	0.8	0.7	0.7	**16**

Total # of 1 dimensional Gaussian r.v.'s simulated = 1.5*10^8

QNET based on 4 degree polynomials, about 2.2 * 10^8

Steady-state Simulation

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- Key idea: Apply TES & simulate

$$\mathbf{X}(\cdot) \;\;\&\;\; \max\{\mathbf{X}(s) : s \geq \cdot\}$$

jointly — NOTE THE INFINITE HORIZON!

Outline

Other Processes subject to TES

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- Long range dependence processes
- Final remarks: Tools applicable to many server problems!

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Conclusion

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- Good Monte Carlo for RBM & fully continuous processes.
- Very powerful tricks...