Rare Event Simulation for Stochastic Networks

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- **Goal of this line of research: To investigate exactly HOW?**

Before I Answer How: Why Would Anybody Care?

A fast computational engine
enhances our ability to quantify uncertainty
via sensitivity analysis & stress tests...

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$$\varepsilon^{-2}\delta^{-1}\mathbf{Com}(n)$$ —> **subexponential complexity in** n

- Total cost $\mathbf{TC}(n) = \mathbf{Com}(n) \times$ *Cost per replication*

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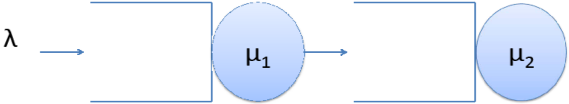
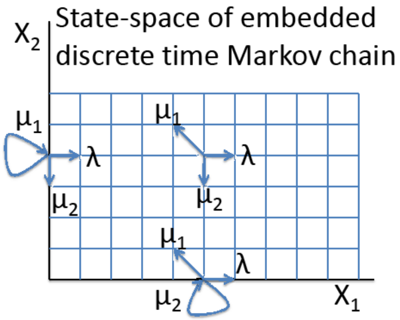
- P_0[Total population reaches n in a busy period]



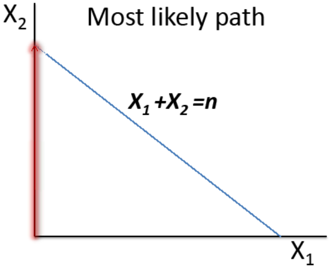
Tandem Jackson networks

- $X_1(k) = \#$ in station 1 at transition k in embedded discrete time Markov chain
- $X_2(k) = \#$ in station 2 at transition k in embedded discrete time Markov chain
- Assume $\lambda + \mu_1 + \mu_2 = 1$



Intuitive sampling & counter-examples

- Assume that $\rho_1 < \rho_2 < 1$ –> 2nd is bottleneck –> $\lambda < \mu_2 < \mu_1$
- Large deviations theory says: **"Most likely path in fluid scale looks like that of system** (μ_2, μ_1, λ)"



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- Importance sampling estimator

$$I\left(T_n < T_{\{0\}}\right) \prod_{k=0}^{T_n-1} \frac{K\left(\left(X_1(k), X_2(k)\right), \left(X_1(k+1), X_2(k+1)\right)\right)}{\widetilde{K}\left(\left(X_1(k), X_2(k)\right), \left(X_1(k+1), X_2(k+1)\right)\right)}$$

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- Reason: Likelihood ratio very poorly behaved when process reaches T_n OUTSIDE most likely region!

Summary of the introduction

- **Large deviations helps computers BUT much more than direct interpretation is needed!**

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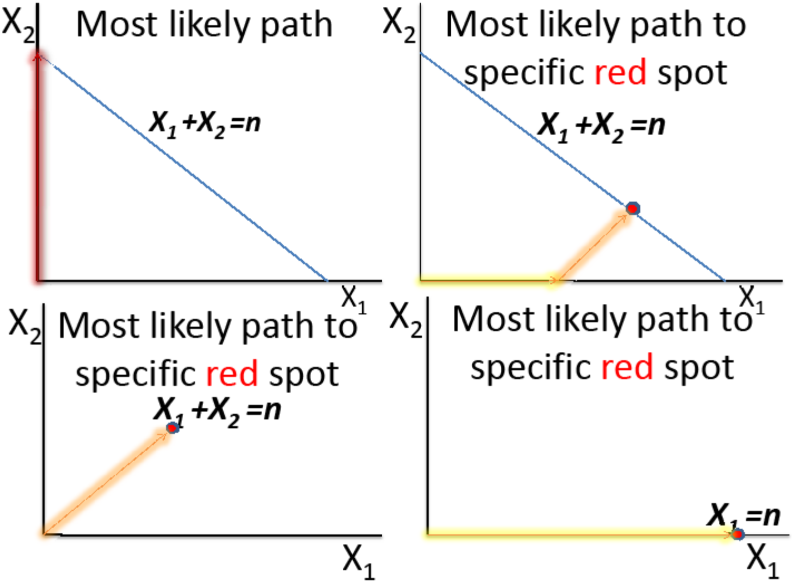
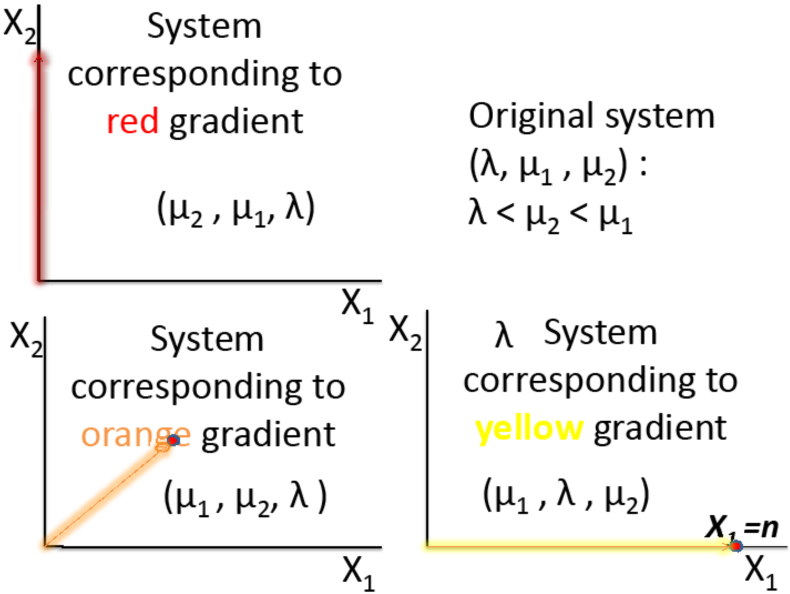
- Note: these results guarantee only **subexponential complexity** (i.e. **Com**$(n) = \exp(o(n))$ replications)

Finitely many gradients



Systems corresponding to gradients



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- NOTE: One applies the same rule EVEN if we are not in the corresponding fluid path!

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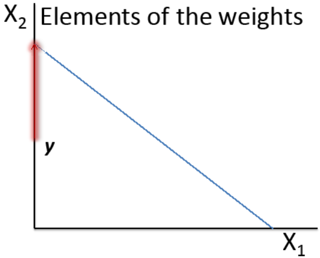

Importance Sampling for Jackson Networks

Theorem (B., Glynn and Leder 2010)

A slight variation of the algorithm by Dupuis, Sezer & Wang '07 satisfies

$$\textit{Total cost} = \mathbf{TC}\,(n) = O\left(n^{2(d+1-\beta)}\right),$$

where $\beta = \#$ *of bottleneck stations.and* $d = \#$ *of stations.*

- **Highlights of proof:**

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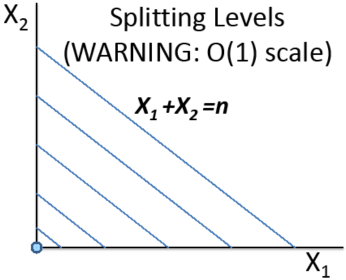
- **Highlights of proof:**
- **A) Translate subsolution into an appropriate Lyapunov inequality**
- **B) Insight into selection of various "mollification" parameters**

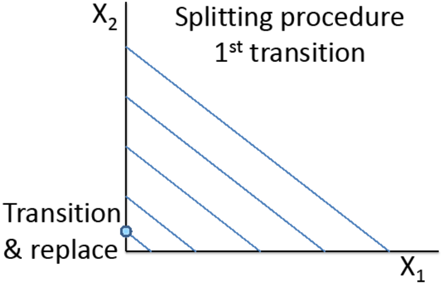
Splitting for Jackson Networks

- Splitting levels



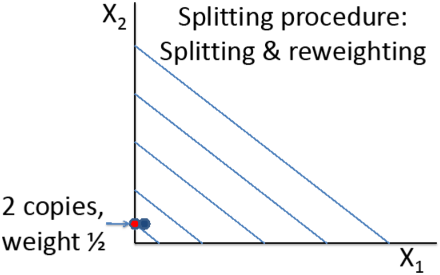
Splitting for Jackson Networks

- First transition



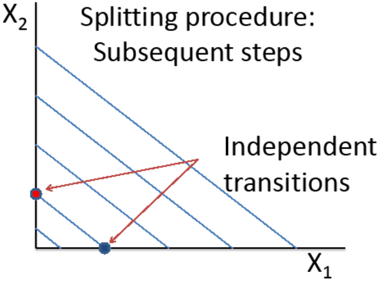
Splitting for Jackson Networks

- Replacement by identical copies & reweighting



Splitting for Jackson Networks

- Subsequent transitions: advancing



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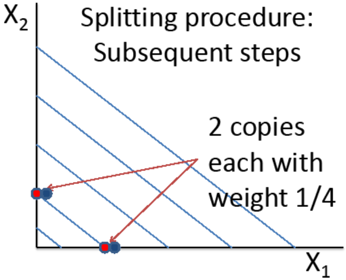
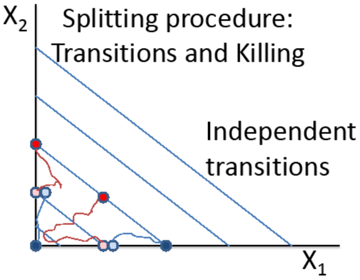


Splitting for Jackson Networks

- One more transition and killing



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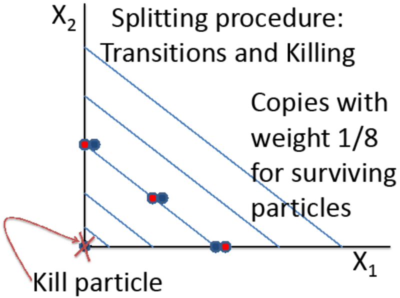

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- $l_n(x) = \#$ of levels to reach total population $= n$

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- Conclusion: Select splitting levels according to level curves of a VERY SPECIFIC linear function

Particles and Large Deviations Analysis

Theorem (B., Leder and Shi '09)

The large deviations splitting rule of Dupuis & Dean '09 has complexity

$$\mathbf{TC}\,(n) = O\left(n^{2\beta+1}\right)$$

$\beta = \#$ *of bottleneck stations.*

- Key idea: understand conditional distribution of network at subsequent milestone events!

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Summary

- **Guaranteed cost of well selected state-dependent samplers (tandem)** $O\left(n^{2(d-\beta+1)}\right)$
- **Guaranteed cost of splitting (general)** $O\left(n^{2\beta+1}\right)$
- Keep in mind that benchmark is solving linear system with $O\left(n^{d}\right)$ unknowns!

$$(Ku)(x) = u(x) \quad \text{subject to} \quad u(x) = 1 \quad x \in A_n$$

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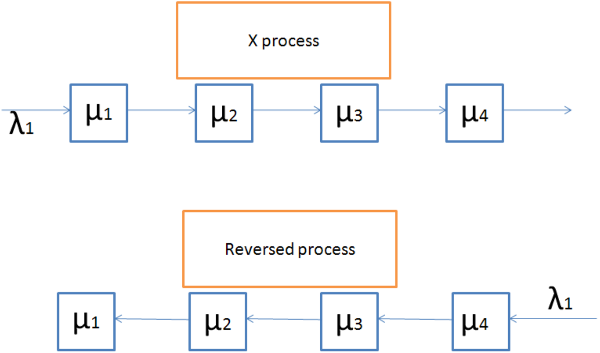
- Time-reversed Markov chain

$$K^{\leftarrow}(y, x) = \frac{\pi(x) K(x, y)}{\pi(y)}$$

Time reversed characteristics for tandem networks

- Time reversed Jackson networks (example tandem)



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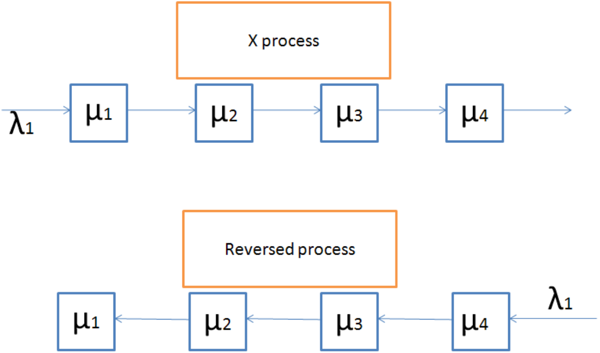

- Steady-state numbers are independent Geometrics with mean $\rho_i/(1-\rho_i)$

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- $T_{\{0\}}$ = 1st return time to origin
- $T\{x\}$ = 1st time to hit x

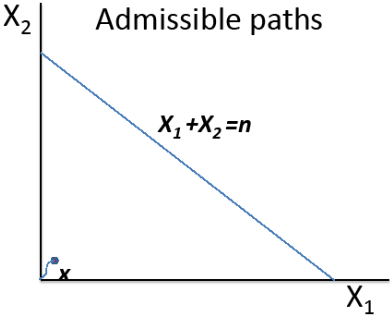
$$P_0\left(T_n < T_{\{0\}}, T_{\{x\}} < T_n \wedge T_{\{0\}}\right)$$

$$= \sum_{\substack{admissible \\ paths:\{x_0 \to x_1 \to \ldots \to x_{T_n}\}}} \{ \underbrace{K\left(x_0, x_1\right)}_{x_0 = 0} \underset{\text{region \& NOT} = x}{\underset{\text{continuation}}{\times \ldots \times}}$$

$$\underbrace{K\left(x_{T_{\{x\}-1}}, x\right) K\left(x, x_{T_{\{x\}+1}}\right)}_{x = x_{T_{\{x\}}}} \underset{\text{region}}{\underset{\text{cont.}}{\times \ldots \times}} \underbrace{K\left(x_{T_n - 1}, x_{T_n}\right)}_{x_{T_n} \in \text{population} = n}$$

Admissible paths

- Section of admissible path reaching x



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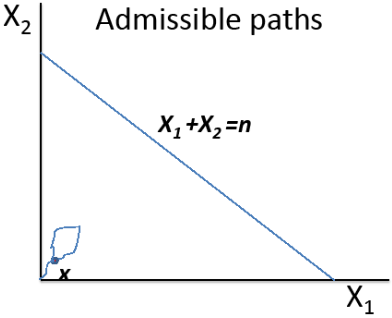

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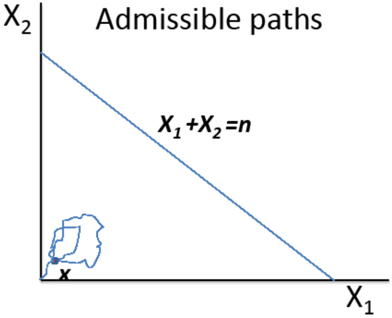

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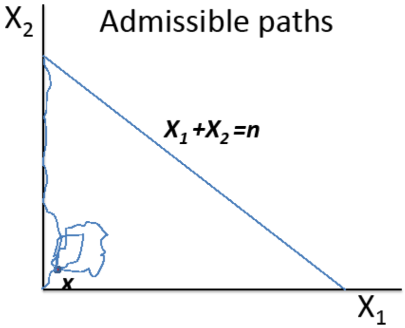

Towards a useful identity

$$
\begin{aligned}
& P_0\left(T_n < T_{\{0\}}, T_{\{x\}} < T_n \wedge T_{\{0\}}\right) \\
= & \frac{1}{\pi(x_0)} \sum_{\substack{\text{admissible} \\ \text{paths}}} \underbrace{\pi\left(x_0\right) K\left(x_0, x_1\right)}_{K^{\leftarrow}\left(x_1, x_0\right) \pi\left(x_1\right)} \times \ldots \times K\left(x_{T_n-1}, x_{T_n}\right) \\
= & \frac{1}{\pi(x_0)} \sum_{\substack{\text{admissible} \\ \text{paths}}} K^{\leftarrow}\left(x_1, x_0\right) \times \ldots \times K^{\leftarrow}\left(x_{T_n-1}, x_{T_n-1}\right) \pi\left(x_{T_n}\right) \\
= & \frac{1}{\pi(x_0)} \sum_{\substack{\text{admissible} \\ \text{paths}}} \pi\left(x_0^{\leftarrow}\right) K^{\leftarrow}\left(x_0^{\leftarrow}, x_1^{\leftarrow}\right) \times \ldots \times K^{\leftarrow}\left(x_{T_n-1}^{\leftarrow}, x_{T_n}^{\leftarrow}\right)
\end{aligned}
$$

- Where $x_k^{\leftarrow} = x_{T_n-k}$ for $k = 0, 1, \ldots, T_n$

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- Admissible starting position

$$y \in D_n := \{y \in population = n \text{ \& } P_y[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow}] > 0\}$$

A useful identity

Lemma

$$
\begin{aligned}
& P_0\left(T_{\{x\}} < T_n \wedge T_{\{0\}}\right) P_x\left(T_n < T_0\right) \\
= \; & P_0\left(T_n < T_{\{0\}}, T_{\{x\}} < T_n \wedge T_{\{0\}}\right) \\
= \; & \frac{1}{\pi(0)} E_\pi[P_{X_0^{\leftarrow}}[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow}, T_{\{x\}}^{\leftarrow} < T_n^{\leftarrow} \wedge T_{\{0\}}^{\leftarrow}] I\left(X_0^{\leftarrow} \in D_n\right)]
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Conclusion:

$$P_x\left(T_n < T_0\right) = \frac{E_\pi[P_{X_0^{\leftarrow}}[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow}, T_{\{x\}}^{\leftarrow} < T_n^{\leftarrow} \wedge T_{\{0\}}^{\leftarrow}]I\left(X_0^{\leftarrow} \in D_n\right)]}{\pi\left(0\right) P_0\left(T_{\{x\}} < T_n \wedge T_{\{0\}}\right)}$$

- Note $P_0\left(T_{\{x\}} < T_n \wedge T_{\{0\}}\right) \geq P_0\left(T_{\{x\}} < T_{\{0\}}\right) > 0$ if $x = O\left(1\right)$.

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- Note $P_0\left(T_{\{x\}} < T_n \wedge T_{\{0\}}\right) \geq P_0\left(T_{\{x\}} < T_{\{0\}}\right) > 0$ if $x = O\left(1\right)$.
- Focus on $x = 0$ for simplicity, then $P_0\left(T_{\{x\}} < T_n \wedge T_{\{0\}}\right) = 1$.

Implications of the identity for simulation

$$P_0\left(T_n < T_0\right) = \frac{E_\pi\left(P_{X_0^{\leftarrow}}\left[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow} | X_0^{\leftarrow} \in D_n\right]\right) P_\pi\left(X_0^{\leftarrow} \in D_n\right)}{\pi\left(0\right)}$$

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General identity for Jackson networks

- How to simulate $\{X_k^{\leftarrow} : k \leq T_0^{-}\}$ on $T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow}$ given $X_0^{\leftarrow} \in D_n$ in $O(n)$ time?

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$$\inf_{y \in D_n} P_y \left[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow} \right] > 0$$

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2. Exists compact set C & $m, a > 0$ such that $\sup_{y \notin C}\left[E_y\left[||X_m^{\leftarrow}||_1 - ||y||_1\right] \leq -a < 0\right.$
3. Use Chernoff's bound and a union bound argument to show that if $||y||_1 = n - K$ for K large enough

$$P_y\left[T_n^{\leftarrow} < T_{\{0\}}^{\leftarrow}\right] \leq \varepsilon$$

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Conclusion:

- $E_\pi(P_{X_0^{\leftarrow}}[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow} | X_0^{\leftarrow} \in D_n])$ is easily estimated if we know how to sample $X_0^{\leftarrow} | X_0^{\leftarrow} \in D_n$ under π

Conditional steady-state distribution

- How to estimate $P_{\pi}\left(X_0^{\leftarrow} \in D_n\right)$ and sample $X_0^{\leftarrow} | X_0^{\leftarrow} \in D_n$ under $\pi\left(\cdot\right)$ in $O\left(1\right)$ time?

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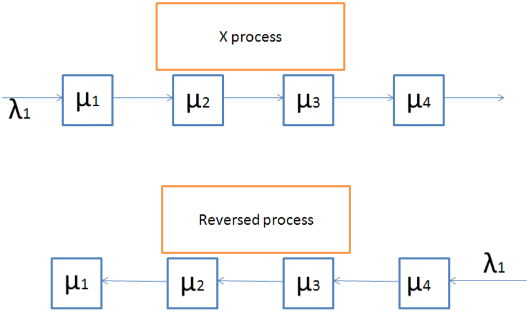

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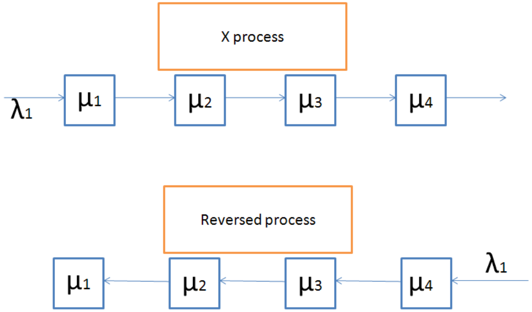

- So concentrate on $X_0^{-}\left(1\right) + \ldots + X_0^{-}\left(d\right) = n -$ *receiving stations*

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- How to estimate $P\left(X_0^-(1)+\ldots+X_0^-(d)=n\right)$, $X_0^-(j)$ independent Geometrics (maybe different parameters)?
- First try: Exponential tilting!
- Well... good (subexponential complexity) but NOT $O(1)$

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Lemma

It is possible to design a sequential importance sampling estimator for

$$P\left(M_1 + \ldots + M_l = n\right)$$

with likelihood ratio $O\left(P\left(M_1 + \ldots + M_l = n\right)\right)$, *where* M_i*'s are independent negative binomial. Thus, one obtains both a strongly efficient estimator and a conditional sampler both with* $O\left(1\right)$ *complexity.*

- Family of samplers: Sort from heavier to lighter tail & use mixtures

$$p\left(i, n-s\right) P\left(M_i = k \middle| M_i \geq n - s_i\right)$$
$$+q\left(i, n-s\right)\right) P\left(M_i = n - s_i - k \middle| M_i \leq n - s_i\right)$$

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$$+q\left(i, n-s\right)) P\left(M_i = n - s_i - k \mid M_i \leq n - s_i\right)$$

- Select probabilities $p\left(i, n-s\right)$ sequentially where $s = M_1 + \ldots + M_{i-1}$

Main result

Theorem (B. 2010)

Given a Jackson network one can estimate

$$P_0\left(T_n < T_0\right)$$

and simulate conditional paths given $T_n < T_0$ *with optimal running time (*$O\left(n\right)$ *complexity).*

Identity

$$\pi\left(0\right)P_0\left(T_n < T_0\right) = E_\pi\left(P_{X_0^{\leftarrow}}\left[T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow} | X_0^{\leftarrow} \in D_n\right]\right)P_\pi\left(X_0^{\leftarrow} \in D_n\right)$$

Event $T_{\{0\}}^{\leftarrow} < T_n^{\leftarrow} | X_0^{\leftarrow} \in D_n$ **is not rare for backward process**
Whole problem is on $X_0^{-}\left(1\right) + \ldots + X_0^{-}\left(d\right) = n$... **plenty of tools!**

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- **Main features of new algorithm appear common to most quasi-reversible queues**

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