Exact Simulation Infinite Server Queue

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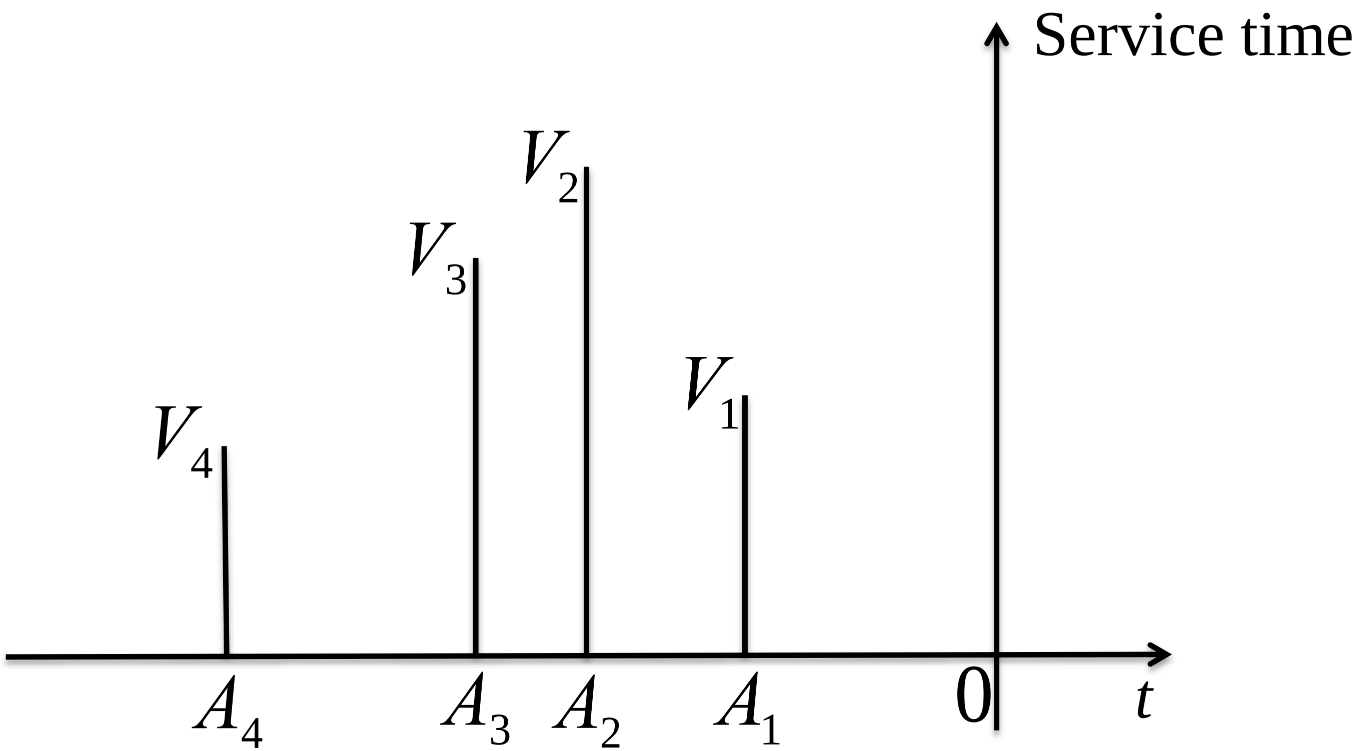
(Joint work with Jing Dong.)

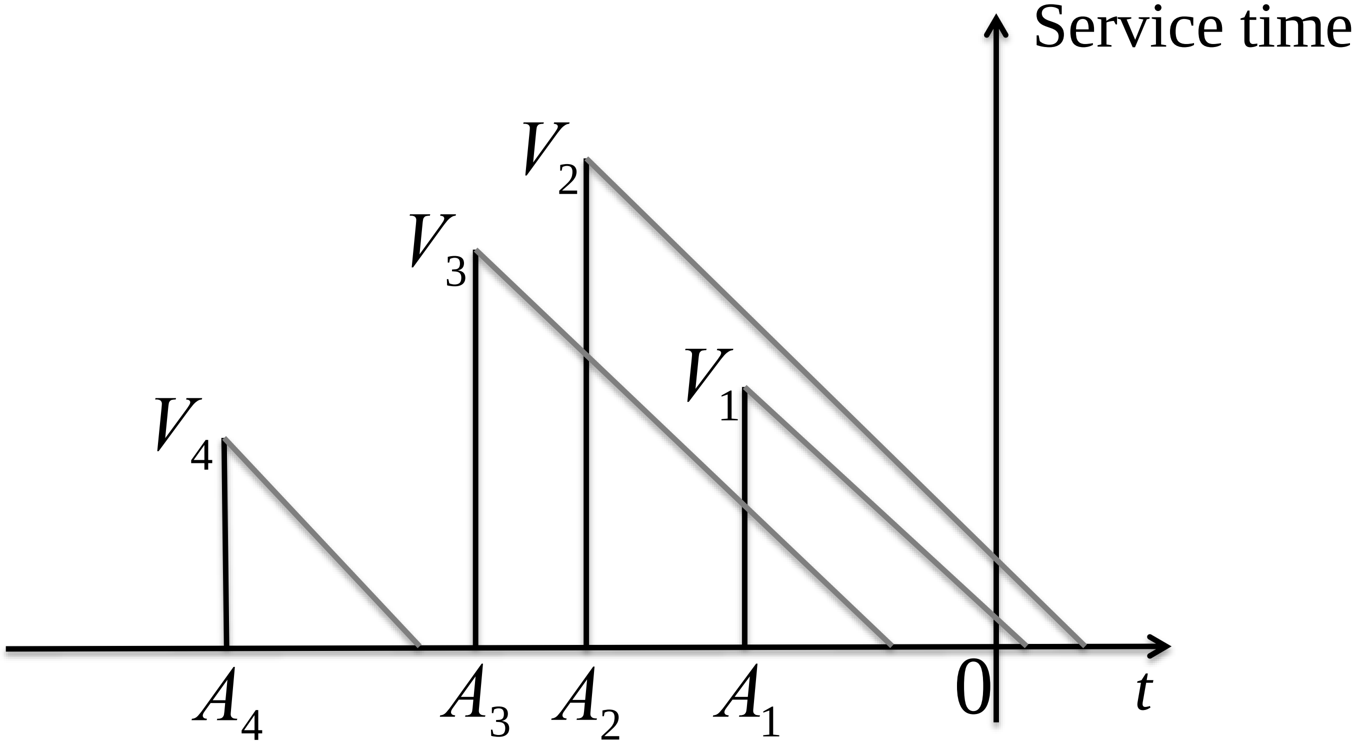
I. Bias Reduction Techniques for Stochastic Networks
   i. Point processes on stable unbounded regions
   ii. Stationary infinite server queue
   iii. Stationary loss systems

Infinite server queue



$$|A_n| = X_1 + X_2 + \cdots + X_n$$

Infinite server queue



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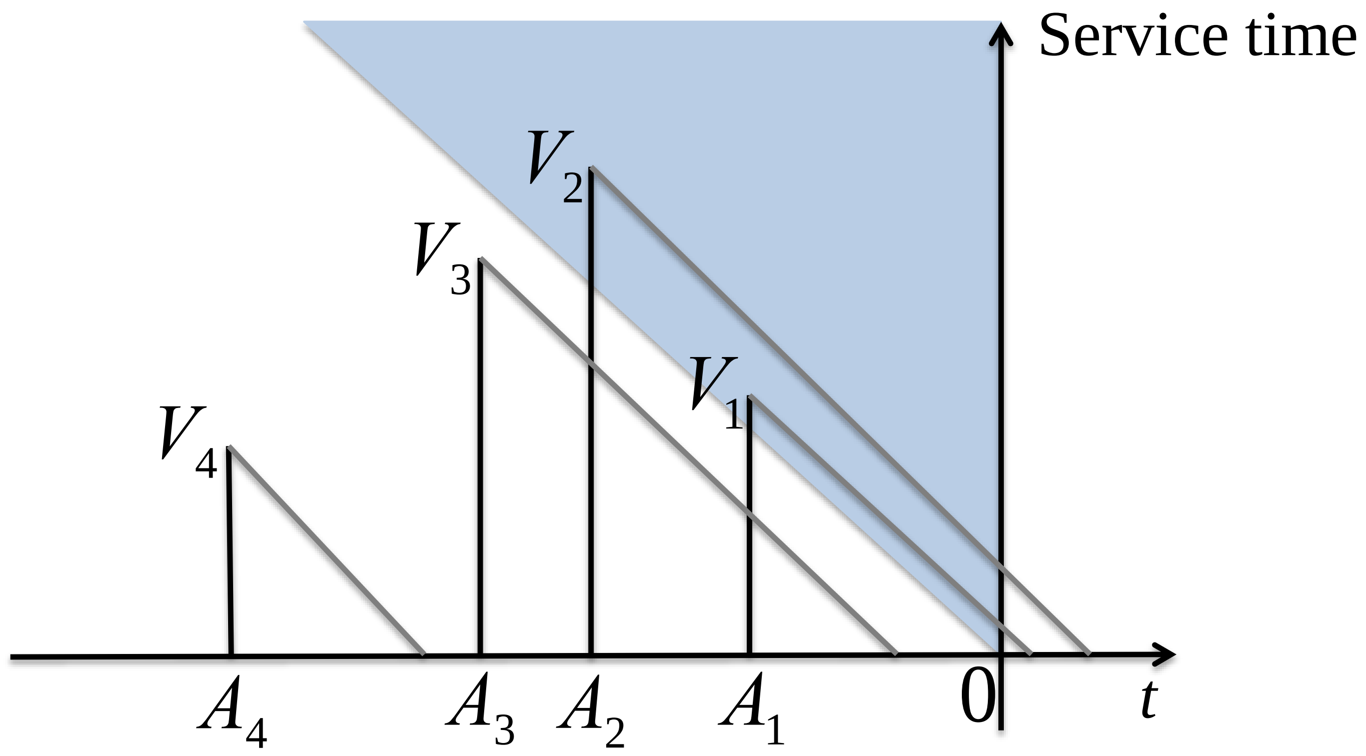
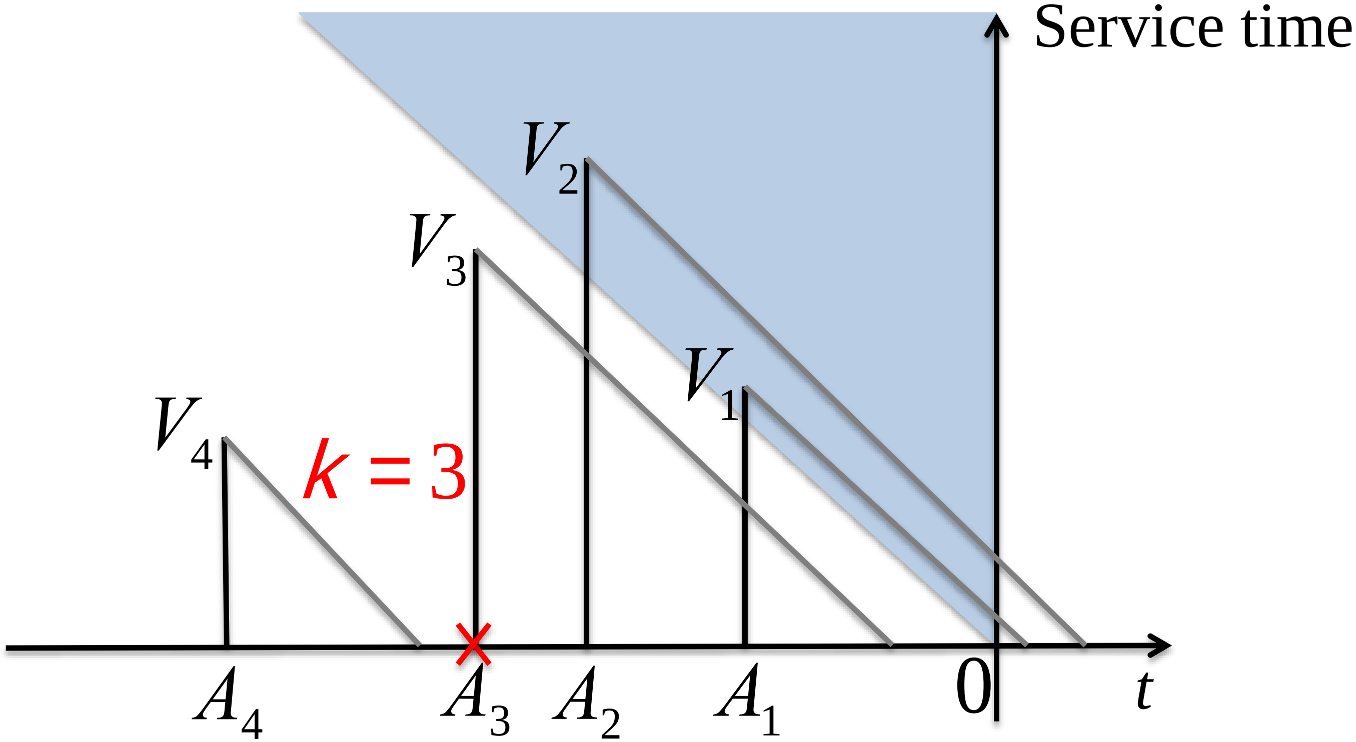


Record Breakers

Look into the future

- Define a sequence of record breakers
- Asking future yes/no questions: are there more record breakers or not
- If yes, find the next record breaker

Infinite server queue



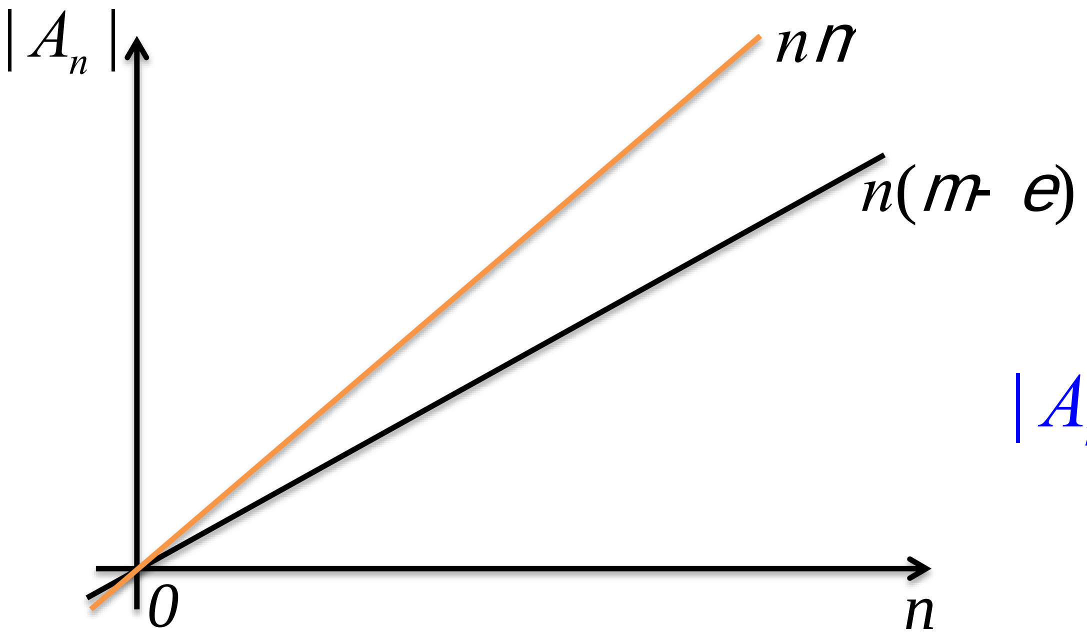
$$k : V_n \le |A_n| \text{ for all } n \ge k$$

Not a stopping time!

Infinite server queue

$E[X_n] = \mu$ and fix $\varepsilon \in (0, \mu)$

$k(A): |A_n| \geq n(\mu - \varepsilon)$ for all $n \geq k(A)$

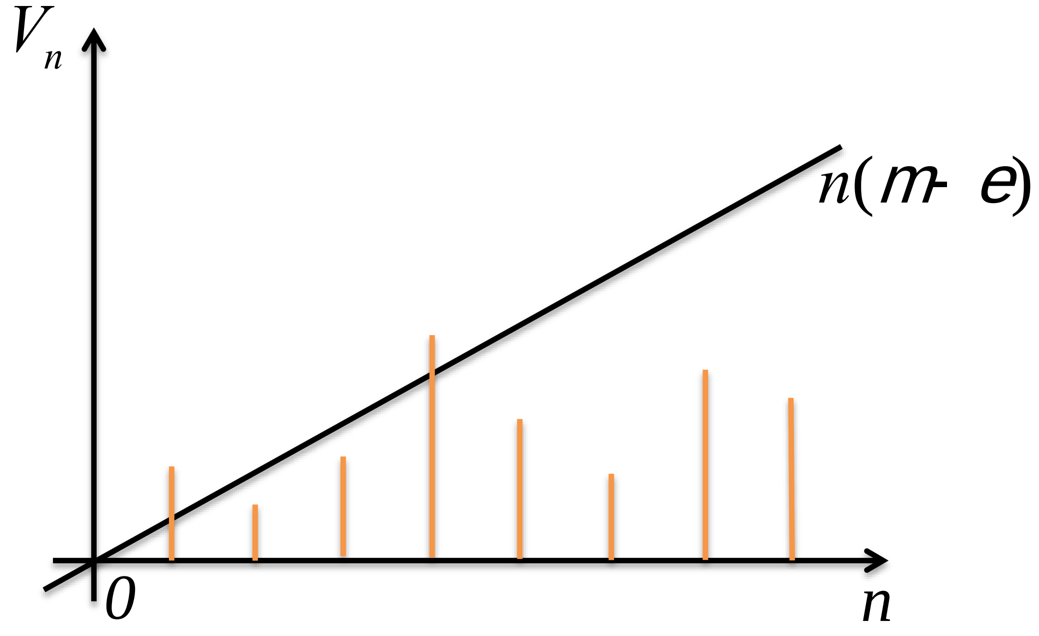


$$|A_n| = \sum_{i=1}^{n} X_i$$

Infinite server queue

$E[X_n] = n\mu$ and fix $\varepsilon \in (0, \mu)$

$\kappa(V) : V_n \le n(\mu - \varepsilon)$ for all $n \ge \kappa(V)$



Infinite server queue

$E[X_n] = \nu$ and fix $\varepsilon \in (0, \mu)$

$$\kappa(A) : |A_n| \geq n(\mu - \varepsilon) \text{ for all } n \geq \kappa(A)$$

$$\kappa(V) : V_n \leq n(\mu - \varepsilon) \text{ for all } n \geq \kappa(V)$$

$$\kappa = \max\{\kappa(V), \kappa(A)\}$$

Infinite server queue: service time process

$\kappa(V): V_n \le n(\mu - \varepsilon)$ for all $n \ge \kappa(V)$

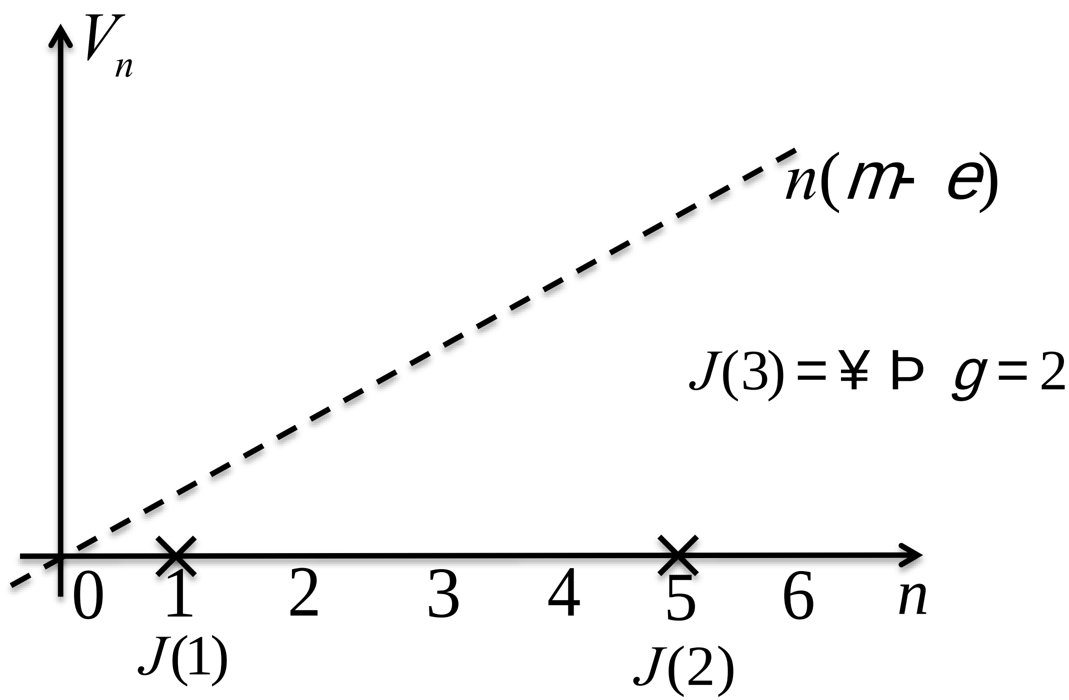
Record breaker: $\{n : V_n > n(\mu - \varepsilon)\}$

J(0):=0

J(l): *l*-th record breaker

$J(\gamma)$: last record breaker

$\Rightarrow \kappa(V) = J(\gamma) + 1$



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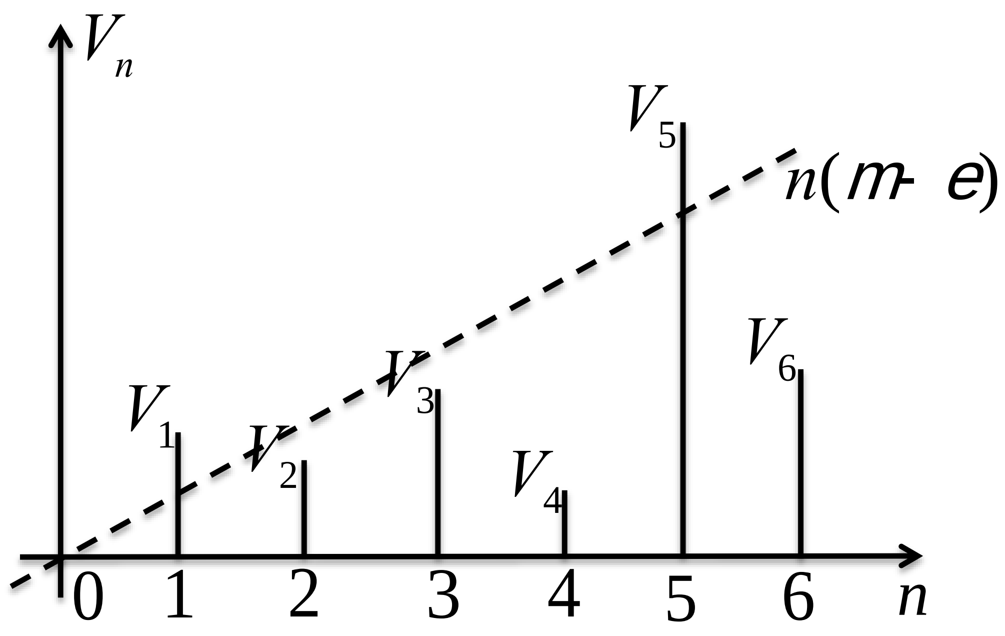


Infinite server queue: service time process

Let $p(n) = P(V_i > n(\mu - \varepsilon))$

Then $P(J(1) = \infty) = \prod_{n=1}^{\infty} (1 - p(n)) > 0$

Upper bound $P(J(1) = \infty) \leq \prod_{n=1}^{h} (1 - p(n)) := u(h)$

Lower bound $P(J(1) = \infty) \geq \prod_{n=1}^{h} (1 - p(n)) g(h) := l(h)$

and $u(h) - u(h-1) = p(h) \prod_{n=1}^{h-1} (1 - p(n)) = P(J(1) = h)$

Negative drifted random walk

$S_n = X_1 + X_2 + \ldots + X_n$ with $E[X_1] < 0$

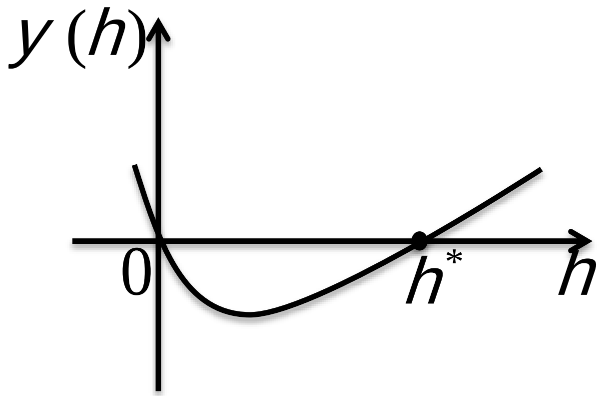
$$T_a = \inf\{n \geq 0 : S_n \geq a\}$$

$$P(T_a = \infty) > 0$$

Negative drifted random walk

$$f_h(y) = \exp(hy - y(h)) f(y) \text{ where } y(h) := \log E\left[\exp(hY_i)\right]$$

$$h^* : y(h^*) := \log E\left[\exp(h^* Y_i)\right] = 0$$



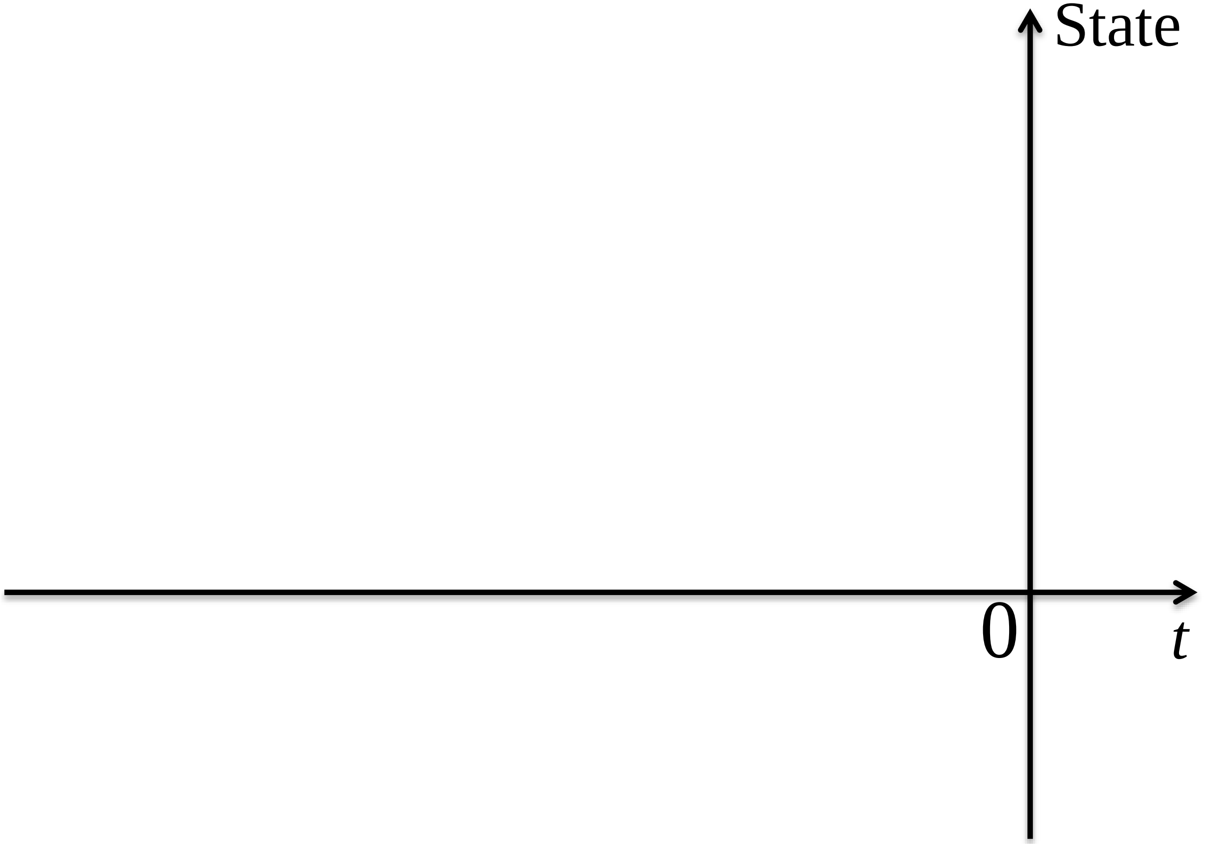
$$f_{h^*}(y) = f(y)\exp(h^* y)$$

$$P(T_a < \infty) = E_{h^*}\left[\exp(-h^* S_{T_a}) 1\{T_a < \infty\}\right] = E_{h^*}\left[\exp(-h^* S_{T_a})\right]$$

$$U \pounds \exp(-h^* S_{T_a})$$

Dominated Coupling from the Past

Dominated Coupling From The Past



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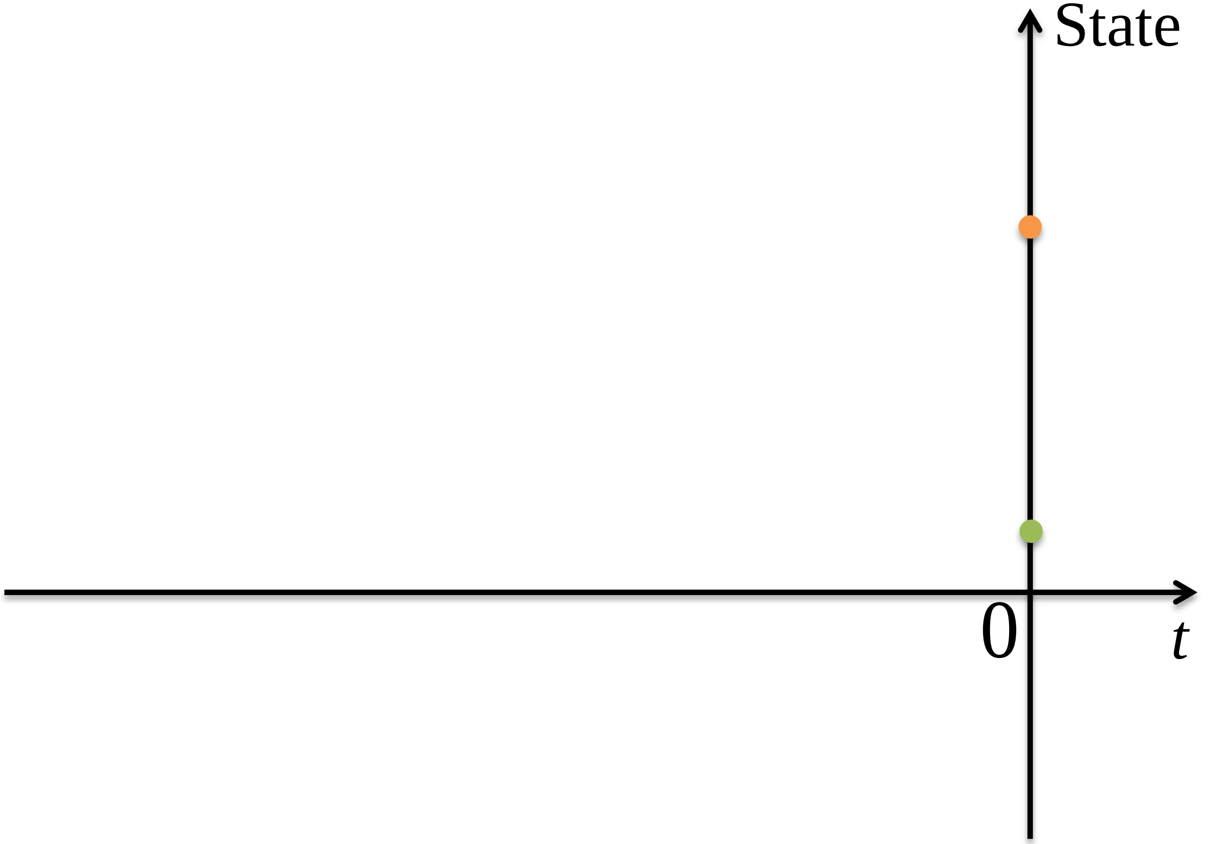

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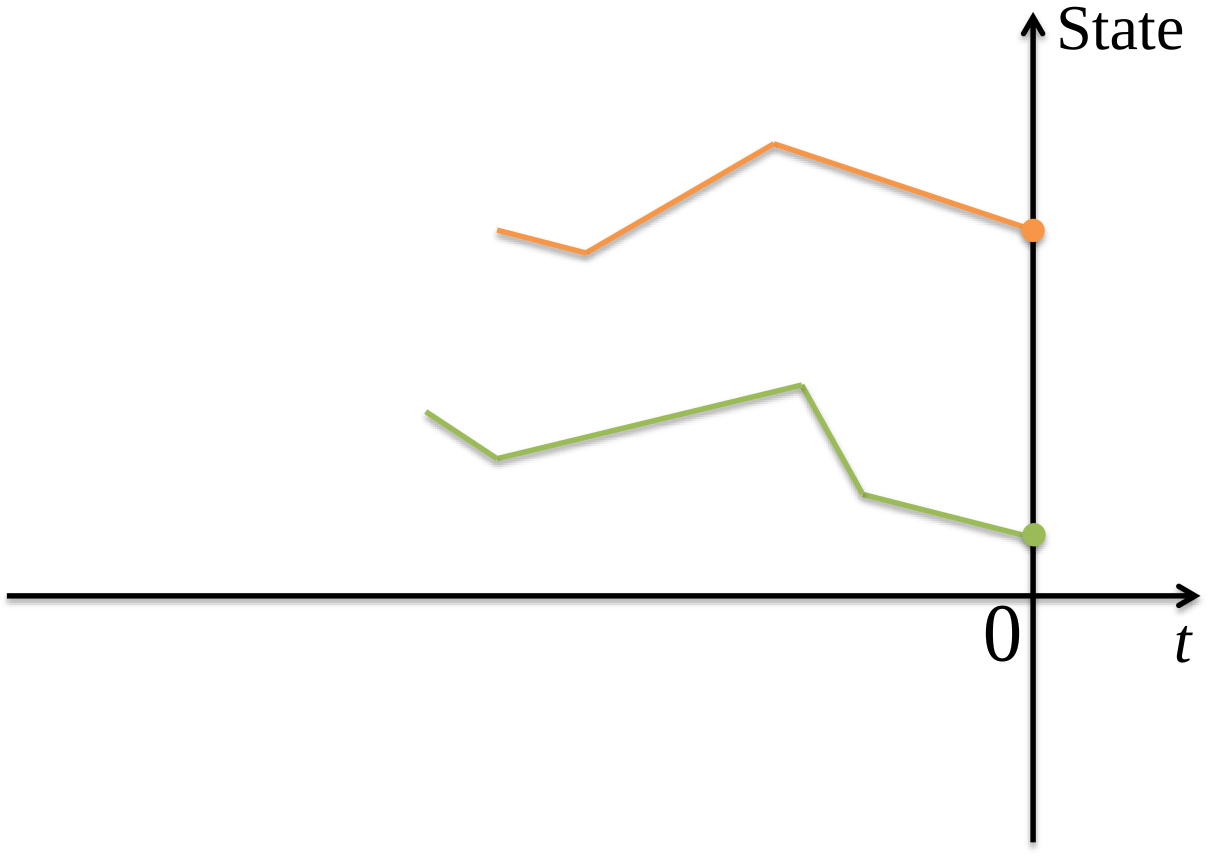

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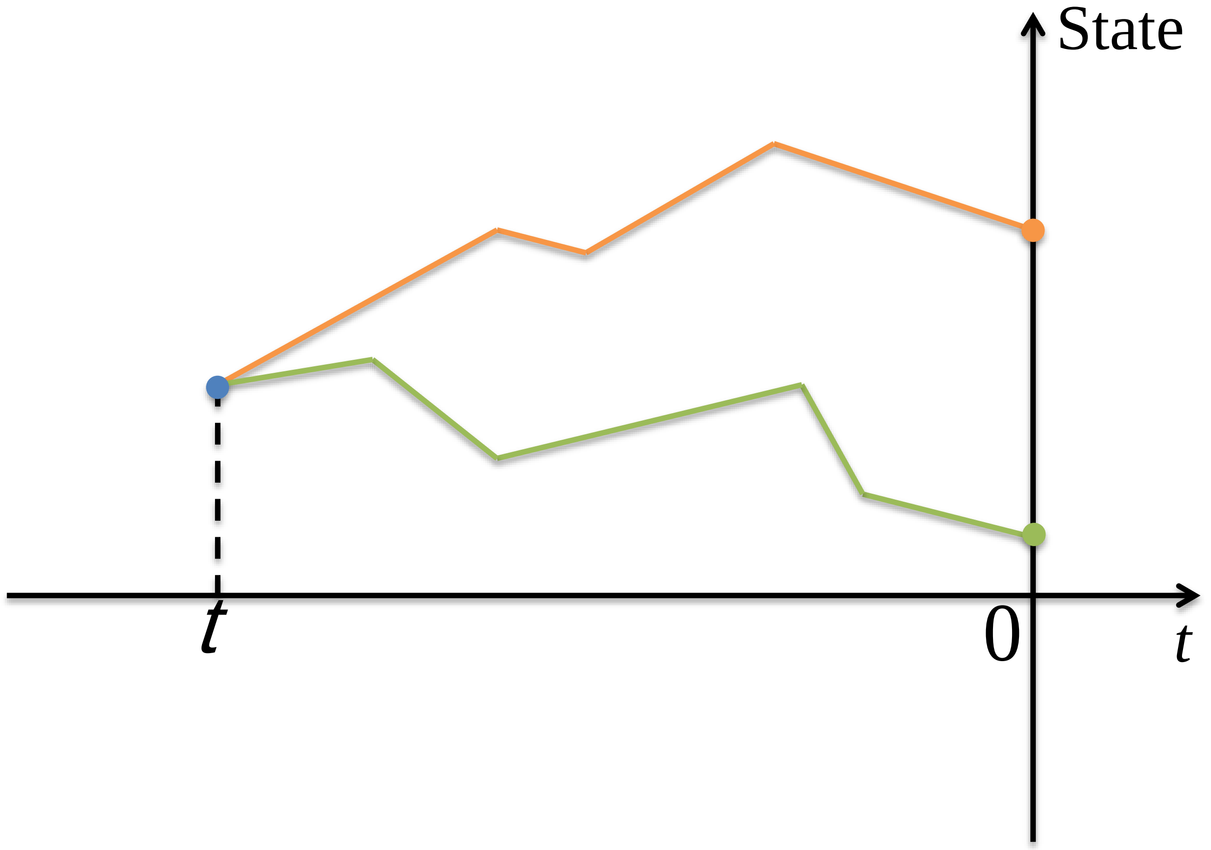

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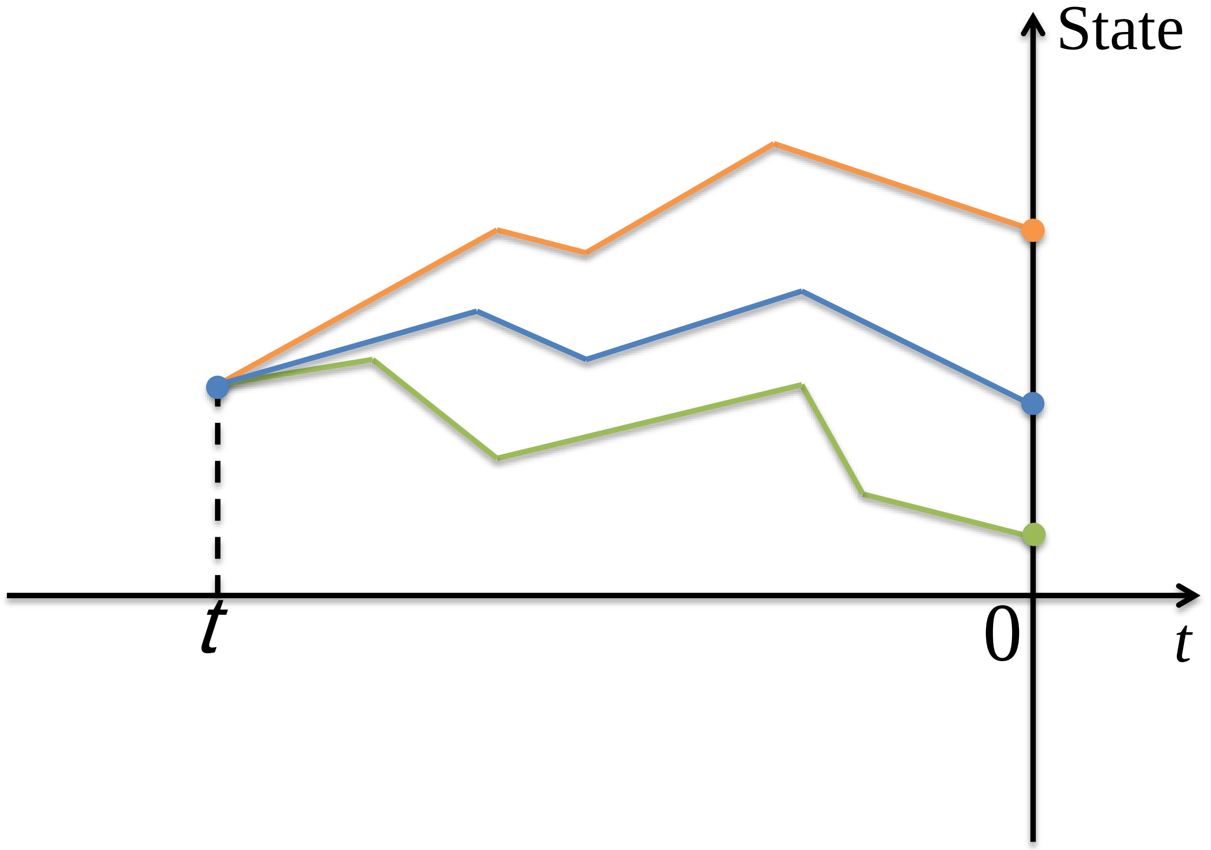
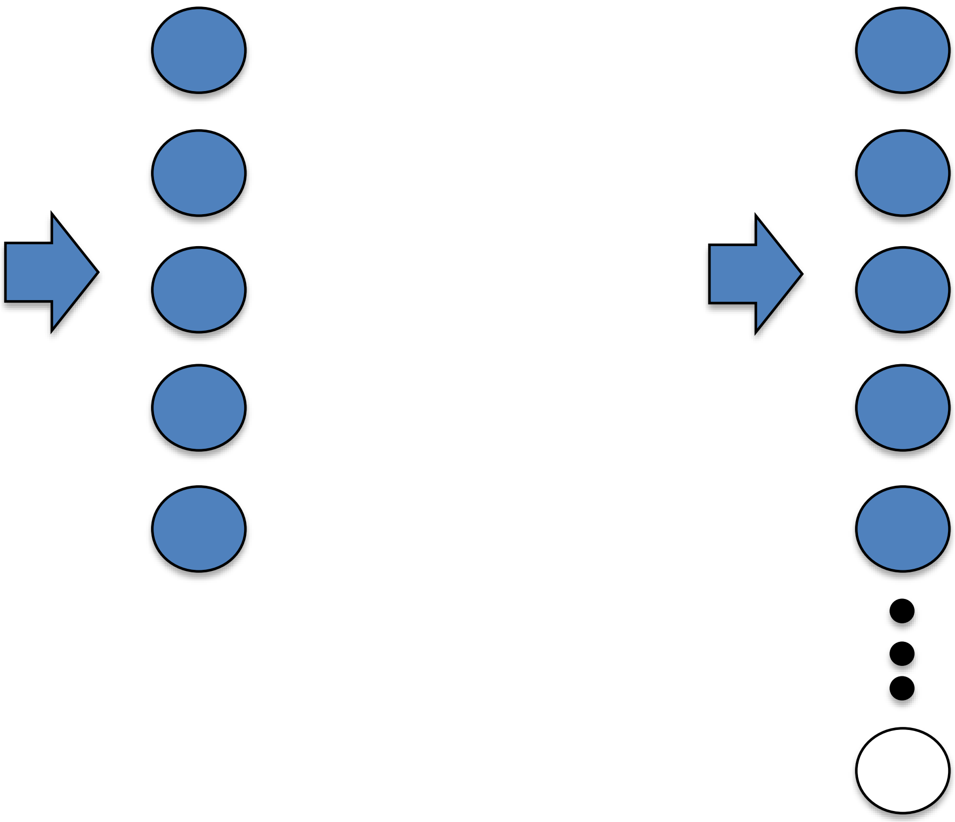


Loss queue: DCFTP

- Bounding process (upper and lower)
  - Upperbound: infinite server queue



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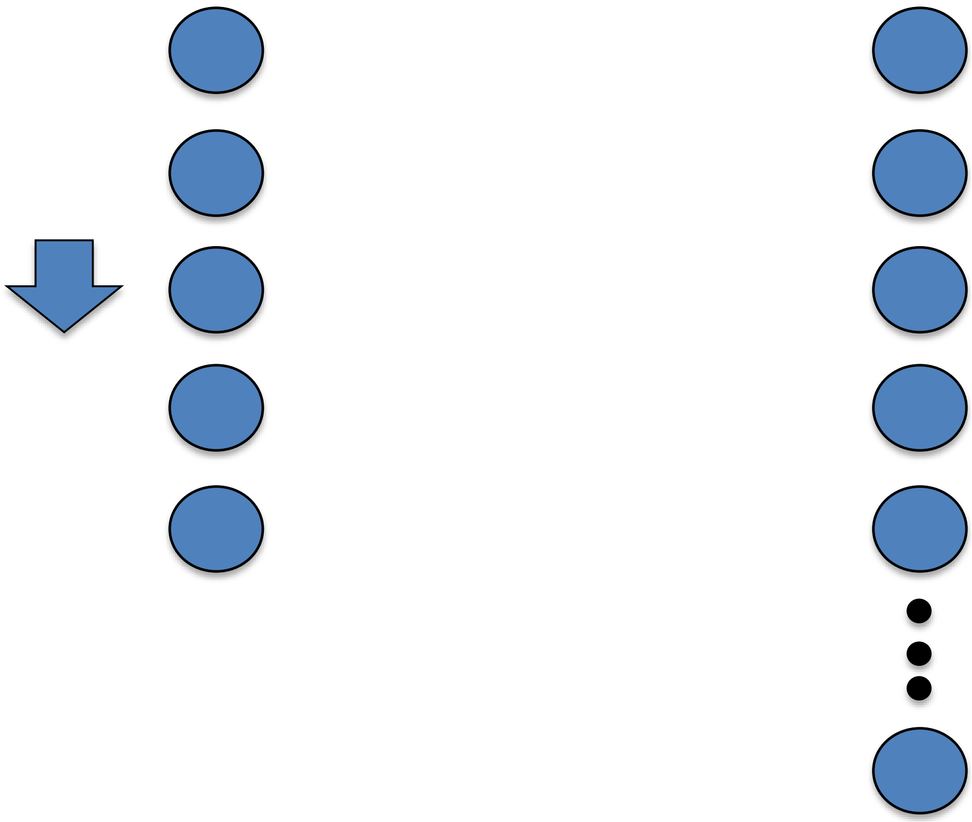

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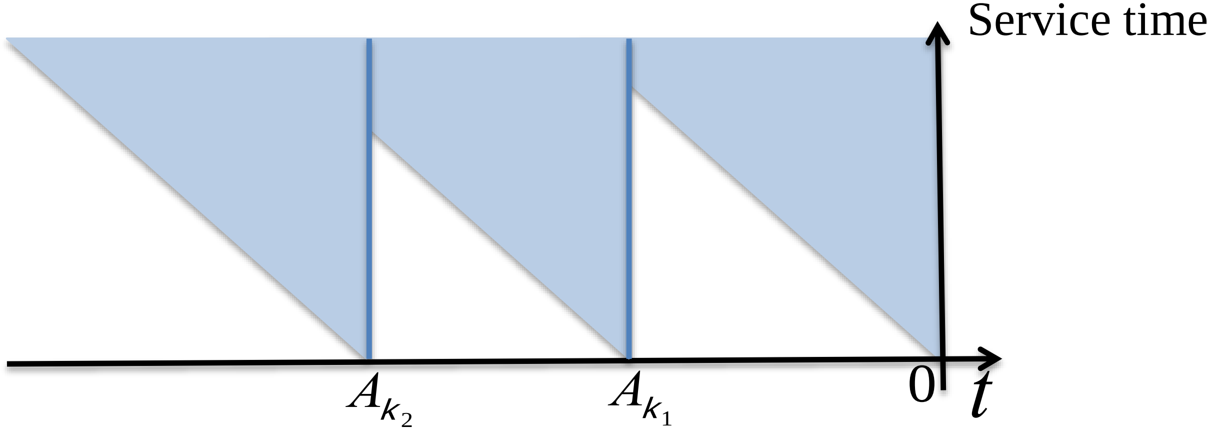
$$Q^L(t,y) \le Q^\infty(t,y) \quad \& \quad E^L(t) = E^\infty(t)$$

$$\Rightarrow \; W^L(t) \le W^\infty(t)$$



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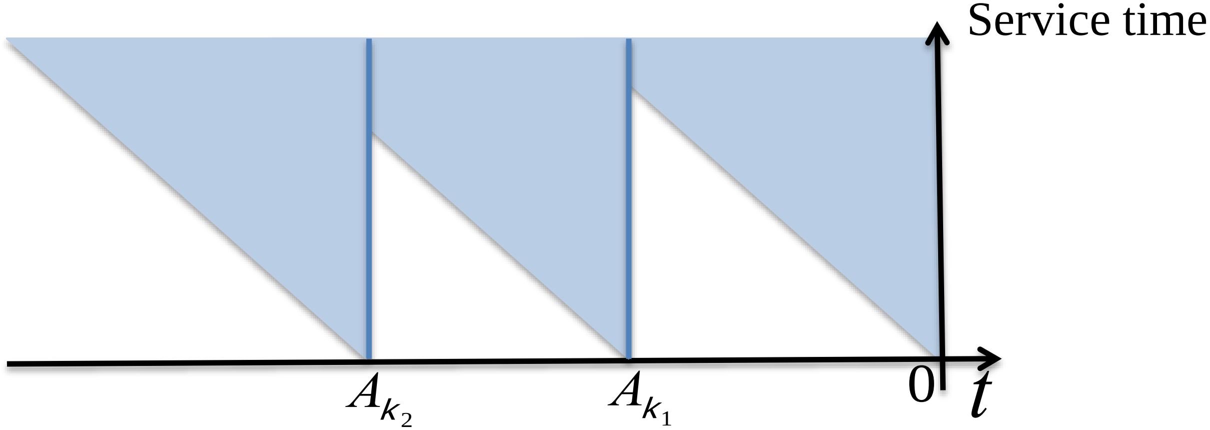

- Lower bound: 0? ➔ not efficient!

Loss queue: DCFTP

- Coalescence time: $t + R^{¥}(t)$

  where $R^{¥}(t)$ largest remaining service time at time t

$$Q^{¥}(t,0) £ C, R^{¥}(t) £ |t|$$

$$\max_{t £ t £ t + R^{¥}(t)} Q^{¥}(t,0) £ C$$