Multiscale Modeling of Order Book Dynamics

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Columbia University

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Outline

1. Goal of the Talk
2. Our Model: Definition and Empirical Validation
3. Price Formation via Queueing Microstructure
4. Cancellation Policy and Continuous Time Dynamics
5. Conclusions

Goal of the Talk

- **Goal: Present and discuss a model for price and the bid-ask spread which:**
  a) Is informed by the full order book dynamics,
  b) It captures key stylized features observed empirically,
  c) Useful in intra-day trading (many minutes / few hours)..

A Picture of a Limit Order Book

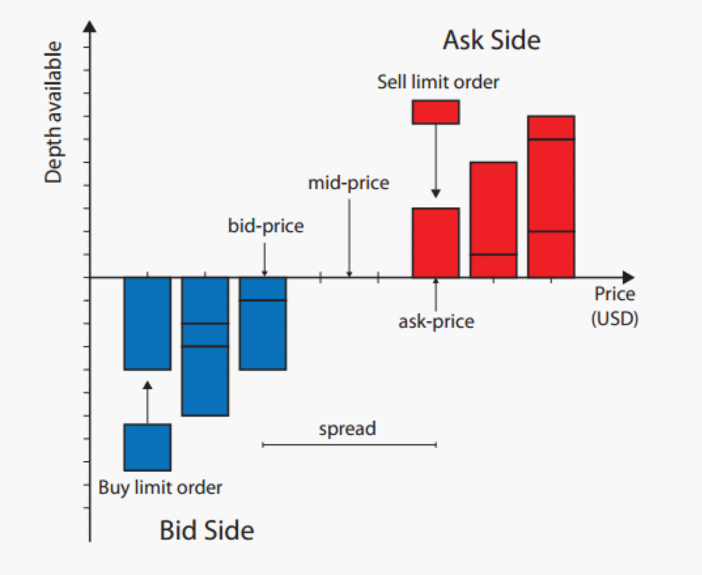

FIG. 1 Schematic of a LOB

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What's the Final Model

- $S(t)$ **= size of bid-ask spread & $M(t)$ = mid-price.**

$$\begin{aligned} dS(t) &= W_{\mu,\sigma}(t) + S(t_-)\,dJ_+(t) + S(t_-)\,dJ_-(t) + dL(t), \\ dM(t) &= \bar{W}_{\bar{\mu},\bar{\sigma}}(t) + S(t_-)\,dJ_+(t) - S(t_-)\,dJ_-(t), \\ S(t)\,dL(t) &= 0, \\ dL(t) &\geq 0. \end{aligned}$$

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- $J_-(\cdot)$ and $J_+(\cdot)$ independent compound Poisson processes with jumps V_- and V_+

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- When do we expect this model to perform well for prices?
- *Answer: When applied to assets with relatively large spread sizes variation (relative to the volatility) over medium time horizon (several minutes, maybe up to a few hours).*
- **How is this model informed by the order book?**

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- Then for $x \geq \varepsilon_0 \geq 0$

$$\Pi_{SELL}\left(x; A\left(t\right), B\left(t\right)\right)^{\gamma_{SELL}} = P\left(V_{+} > x/S\left(t\right)\right).$$

and

$$\Pi_{BUY}\left(x; A\left(t\right), B\left(t\right)\right)^{\gamma_{BUY}} = P\left(V_{-} > x/S\left(t\right)\right).$$

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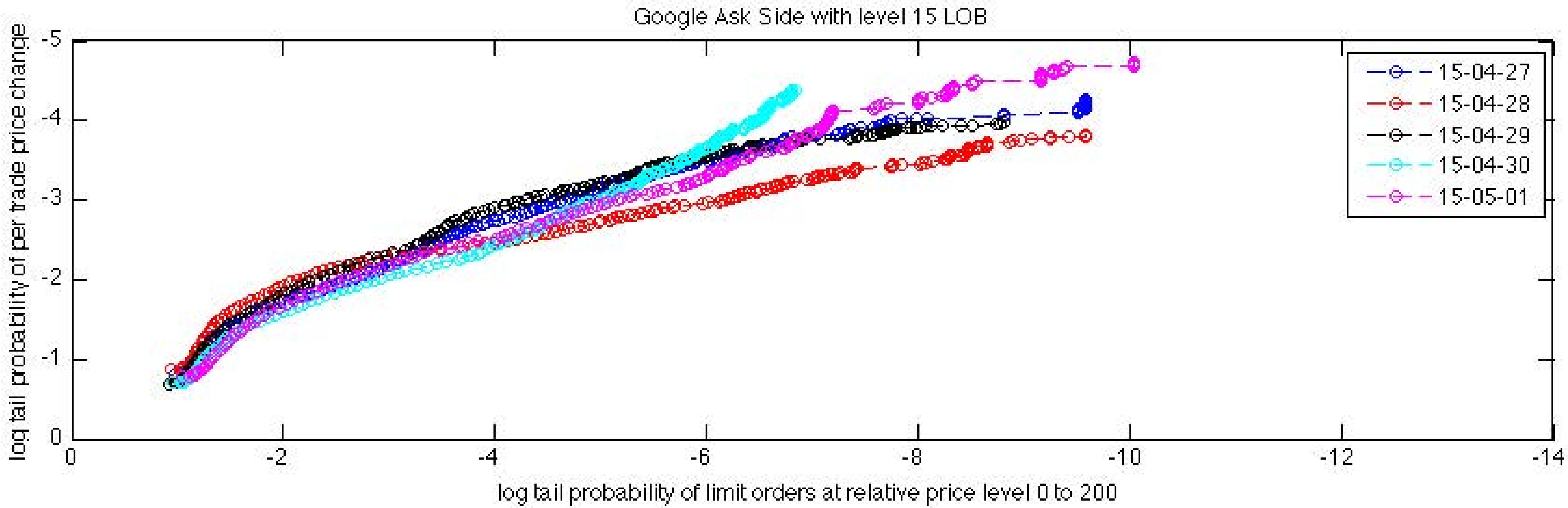
- **Outcome should look like a straigh line!**

Some More Empirical Analysis

Pictures here...


Google Ask Side with level 15 LOB
log tail probability of per trade price change
log tail probability of limit orders at relative price level 0 to 200
-5
-4
-3
-2
-1
0
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-12
-14
15-04-27
15-04-28
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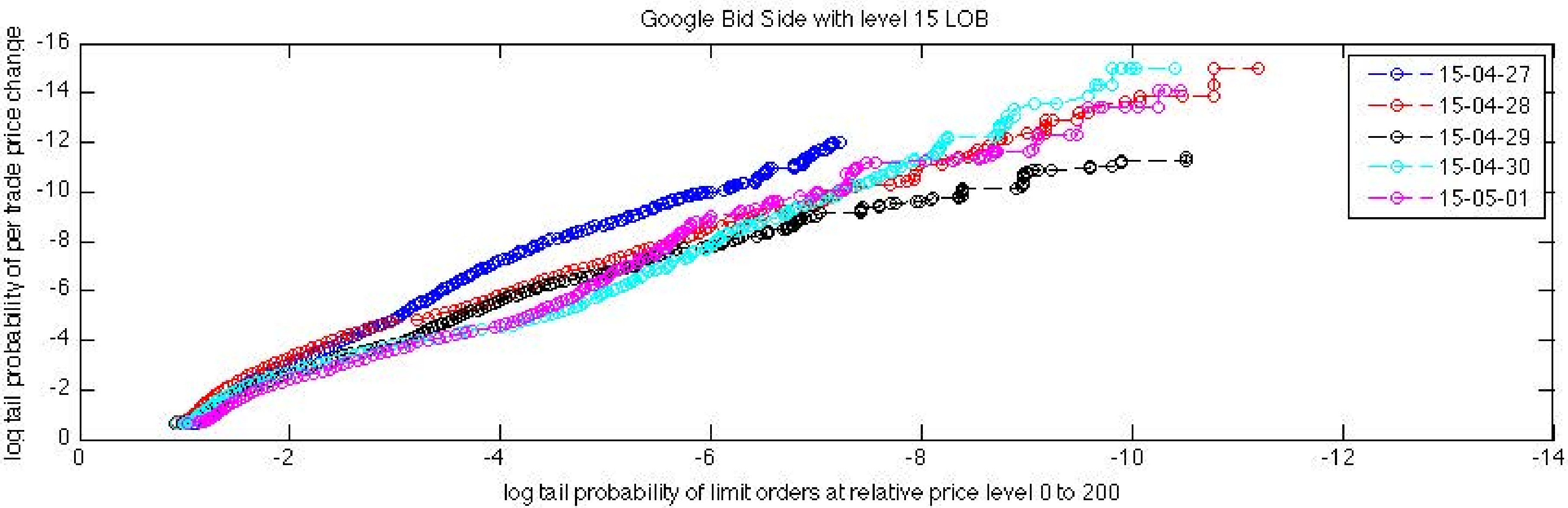
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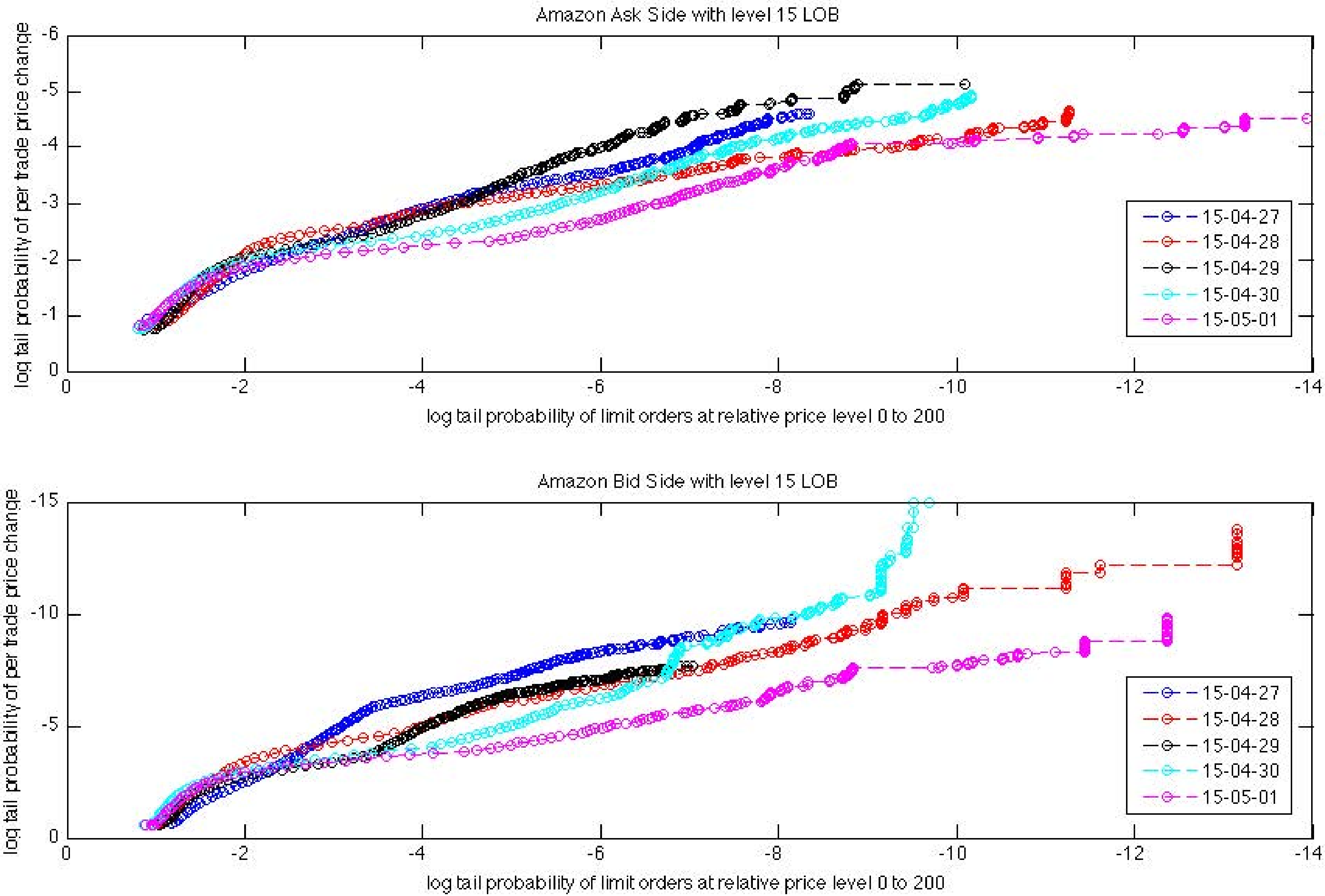
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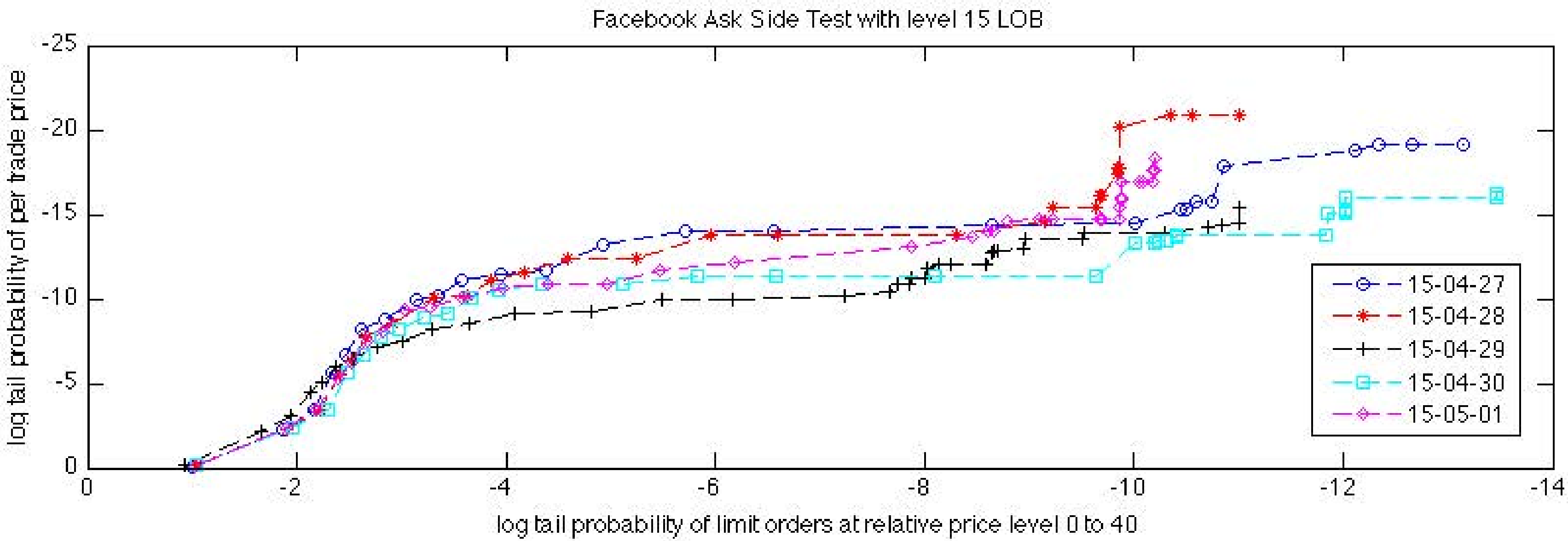
Facebook Ask Side Test with level 15 LOB
log tail probability of per trade price
log tail probability of limit orders at relative price level 0 to 40
-25
-20
-15
-10
-5
0
0
-2
-4
-6
-8
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-12
-14
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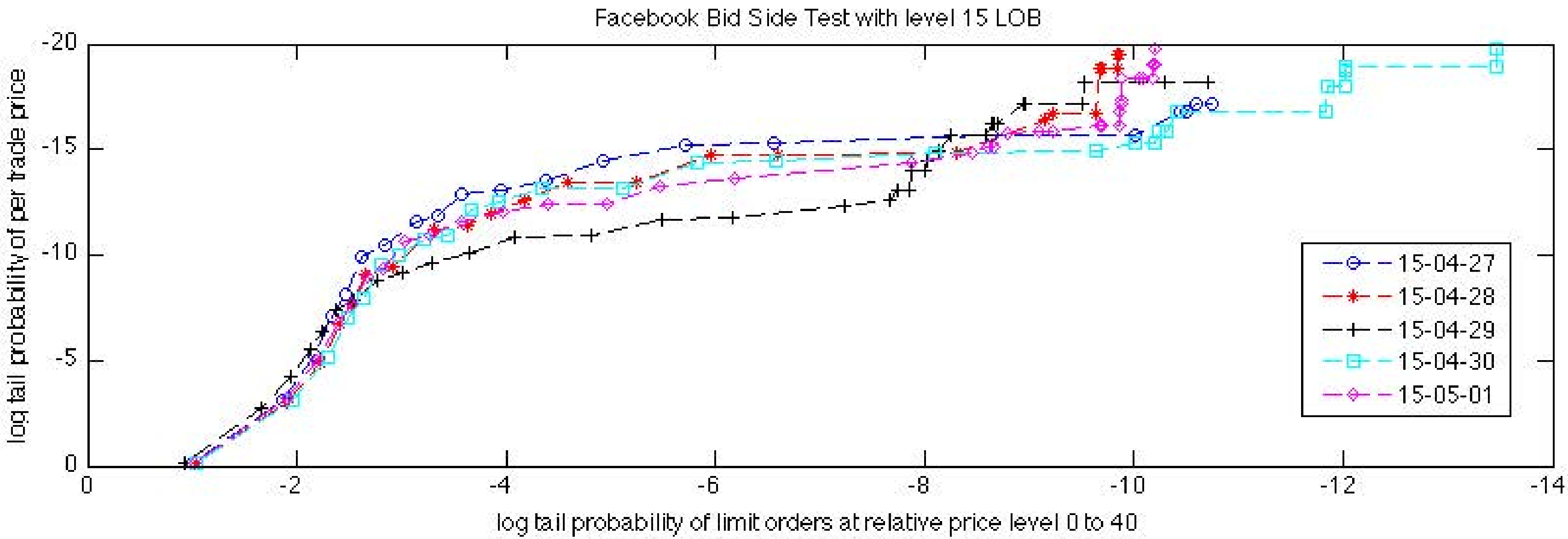
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Warning: Don't Use Model for Small Spread-Size Stocks

Avg. Spread Size (in $ cents)	4/27	4/28	4/29	4/30	5/1
Google	28.69	27.07	34.11	30.39	27.23
Facebook	1.43	1.41	1.68	1.42	1.45
Amazon	16.22	14.86	21.95	20.58	17.23

Important Implications

Bid-Ask processes only encode lots of info from full order book!

Full order book directly feeds the dynamics of Bid-Ask processes!

This whole encoding obeys relatively simple statistical rules!
(Proportional hazards.)

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  3. Understand why so much information can be decoded from prices only?

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4. No cross over of limit orders to opposite side of mid-price.
5. Each order at relative price $i\delta$ cancels at rate $\alpha_{BUY}\left(i\delta, \bar{A}\left(t_k\right), \bar{B}\left(t_k\right)\right)$ or $\alpha_{SELL}\left(i\delta, \bar{A}\left(t_k\right), \bar{B}\left(t_k\right)\right)$.

Discussion About Assumptions

1. Arrival Limit Orders $= \lambda_n >> \mu_n =$ Arrival Market Orders:

Google	4/27/2015	4/28/2015	4/29/2015	4/30/2015	5/1/2015
Total Limit Orders	80,537	78,944	77,215	100,798	66,238
Total Market Orders	9,412	8,016	5,868	8,505	7,038
Facebook					
Total Limit Orders	442,425	483,338	489,886	472,251	363,833
Total Market Orders	31,973	37,378	29,456	36,558	30,455
Amazon					
Total Limit Orders	100,263	131,648	125,555	162,561	123,127
Total Market Orders	13,148	14,804	9,225	11,344	10,094

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- **A4:** No cross over of limit orders to opposite side of mid-price.

Proportion of Cross Limit Orders	**4/27**	**4/28**	**4/29**	**4/30**	**5/1**
Google	2.12%	1.77%	2.31%	1.79%	1.84%
Facebook	2.82%	3.80%	4.77%	3.40%	3.01%
Amazon	2.55%	2.47%	2.45%	1.57%	1.94%

Discussion About Assumptions

- **A5:** $\alpha_{BUY}\left(i\delta, \bar{A}\left(t_k\right), \bar{B}\left(t_k\right)\right)$ cancellation rate $i\delta$ PER order not standard in literature BUT this makes sense...

  Whole system is a coupled multiclass two server queuing network.

Averaging Principle Between Market Order Arrivals

Theorem

(B., Chen & Pei) At arrival of $(k+1)$-th market order the order book follows the distribution of independent $M/M/\infty$ queueing systems the i-th with parameter

$$\rho_n(i) = \lambda_n \frac{p(i\delta, A(t_k), B(t_k))}{\alpha(i\delta, A(t_k), B(t_k))} \cdot \delta.$$

Recall: Steady-state number in system of $M/M/\infty$ is Poisson with parameter $\rho_n(i)$.

Proof.

Averaging principle for Martingale problems (see Kurtz (1992)). □

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How to Pick a Cancellation Policy that Explains Empirical Findings?

- **Assumption (C):** For $x > x_0$

$$\alpha(x, A, B) \approx \gamma \times \left(1 - \bar{\Pi}(x, A, B)\right).$$

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- Tends to increase as depth increases (very reasonable).
- If $\gamma \approx 1$ one can argue that, the equilibrium rate of execution, is

$$\frac{\mu\theta(i\delta, A, B)}{\mu\theta(i\delta, A, B) + \alpha(i\delta, A, B)} \approx \frac{\mu}{1+\mu}$$

is constant at any level.

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- So, **auxiliary increment distribution:=** $\Delta_{k+1}(S(t_k))$ depending on spread:

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$$\begin{aligned} \Delta_{k+1}(S(t_k)) &= (-1)^{R_{k+1}}(1 - I_{k+1})\,\delta_n \left[U_{k+1}/\left(n^{1/2}\delta_n\right)\right] \\ &\quad + I_{k+1}[S(t_k)\,V_k/\delta_n]\delta_n. \end{aligned}$$

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- Second piece captures the structural part, namely, the fact that for $x > x_0$

$$\Pi(x; A, B)^{\gamma} \approx \theta(x, A, B).$$

Why Local Times in Limiting Process?

- It turns out that in terms of auxiliary increment distribution $\Delta_{k+1}(S(t_k))$, one gets

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- We use techniques from diffusion approximations and continuity of the so-called Skorokhod map to establish the result.

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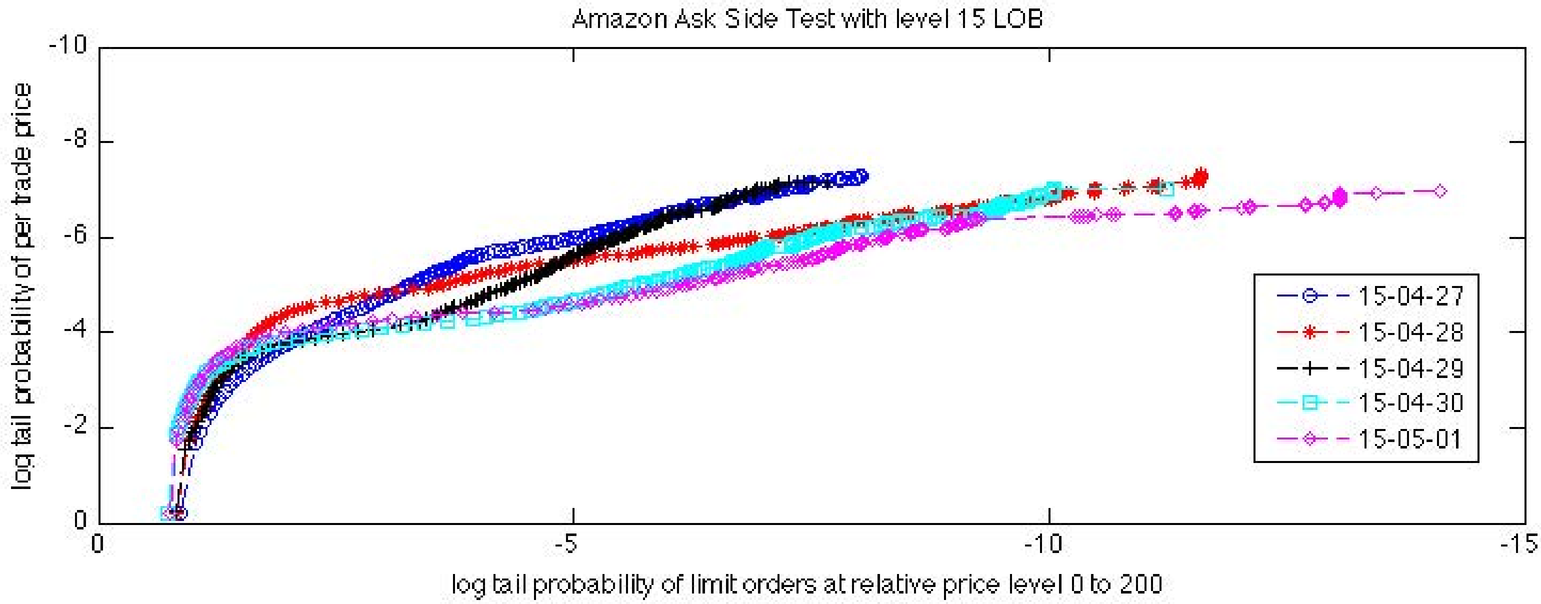

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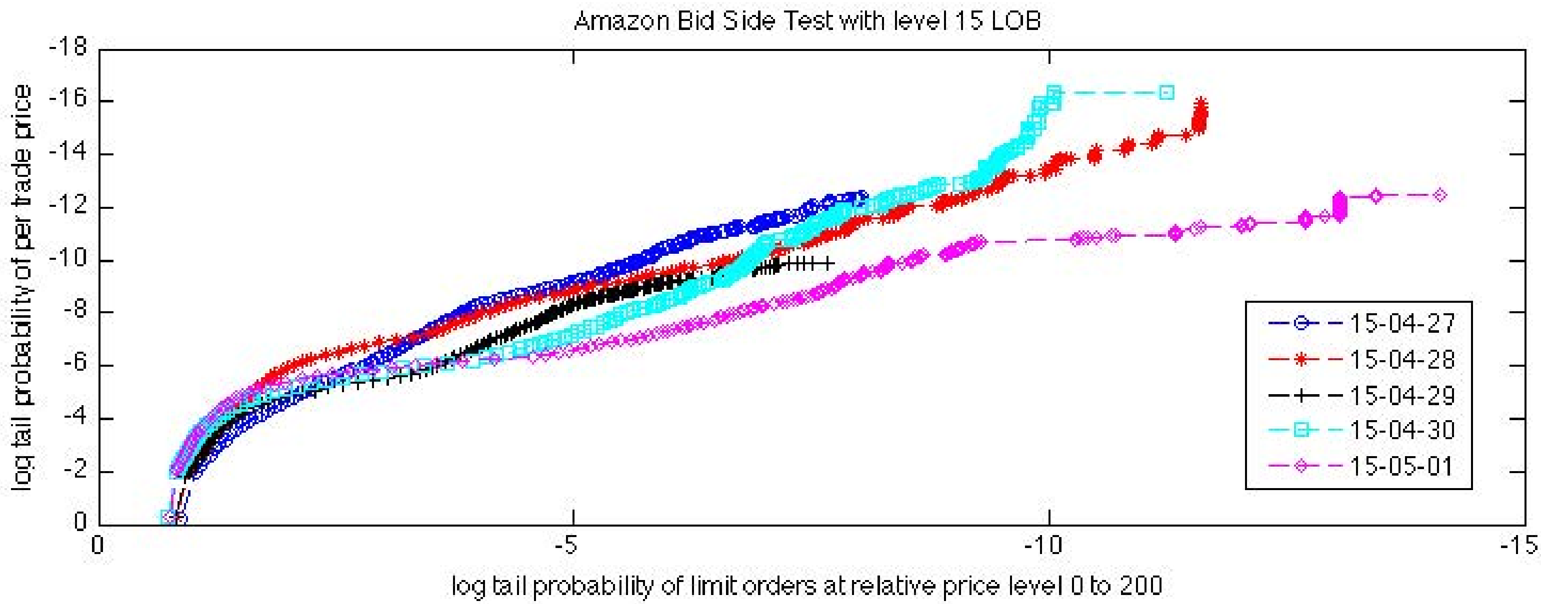

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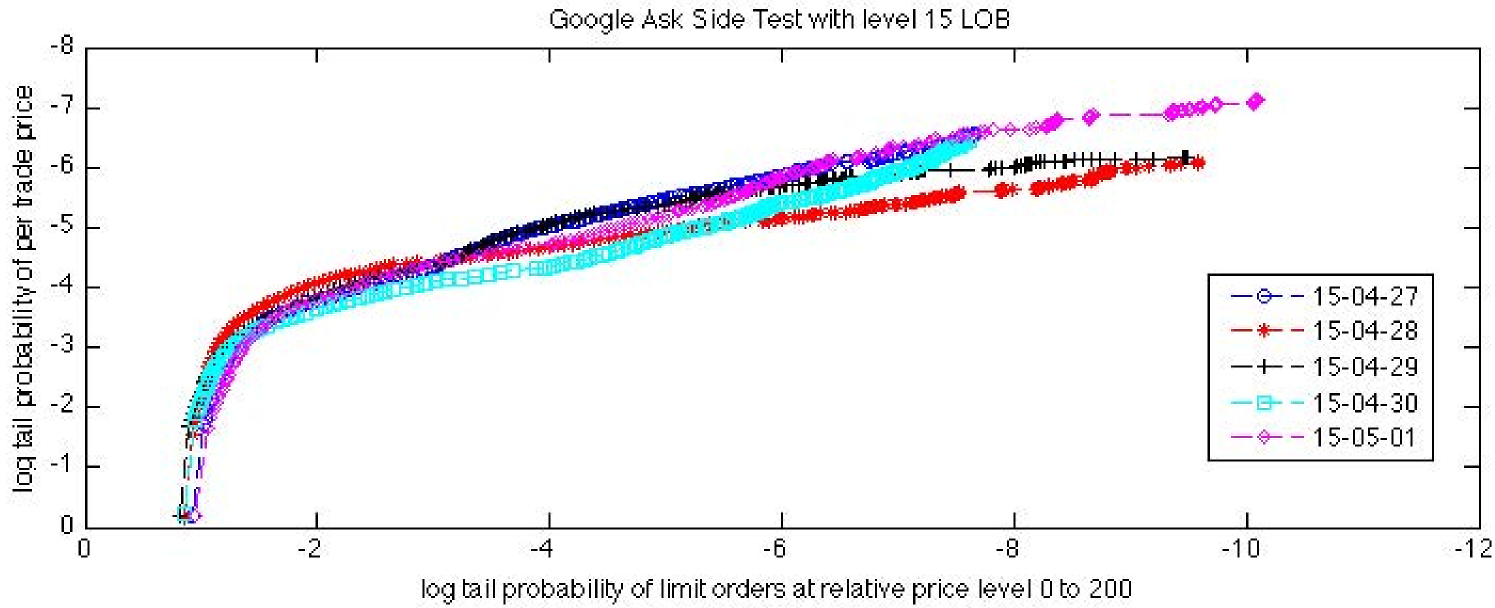
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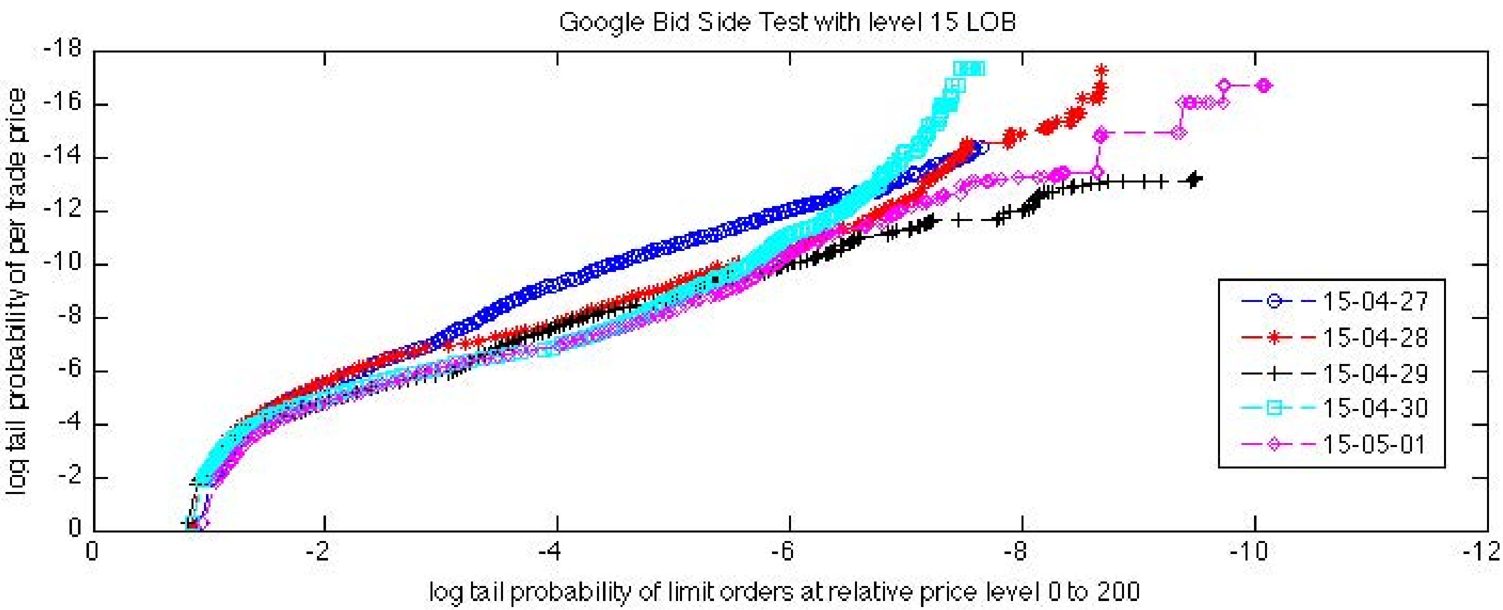
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