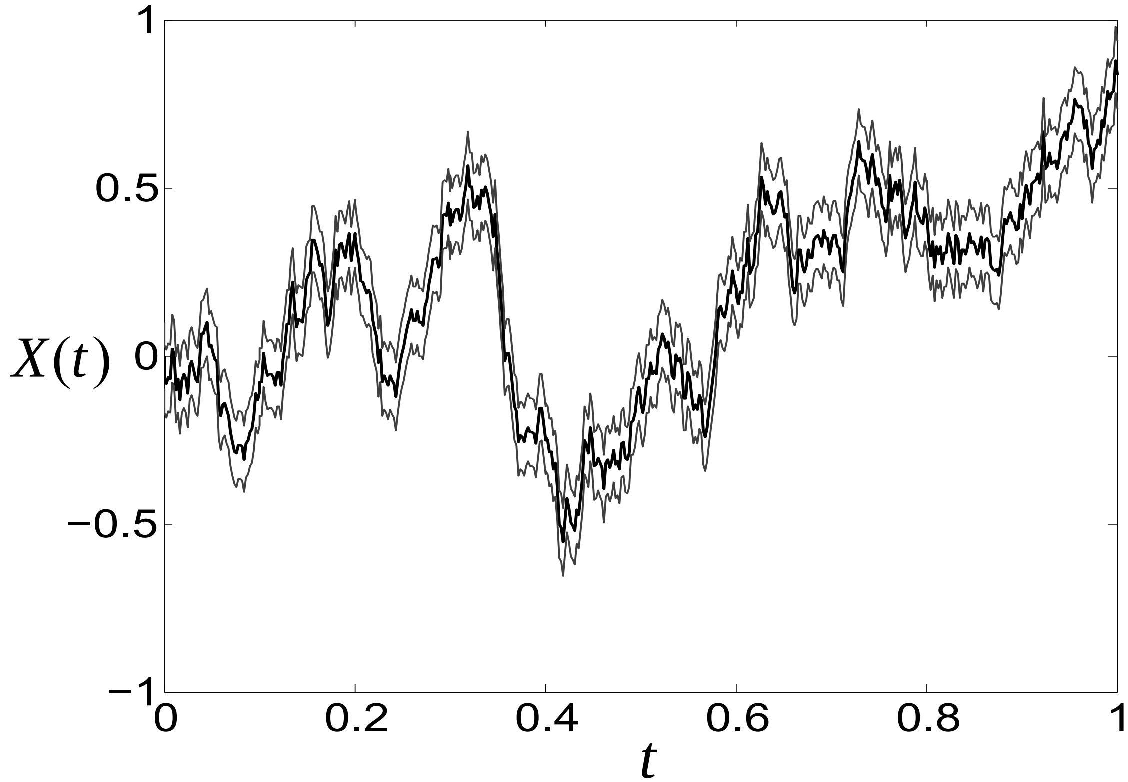
ε-Strong Simulation for Multidimensional Stochastic Differential Equations via Rough Path

ε-Strong Simulation for SDEs

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t)$$

A piecewise constant path, X_ε, such that

$$\sup_{0\le t\le T} |X_\varepsilon(t) - X(t)| \le \varepsilon \ \ w.p.1$$

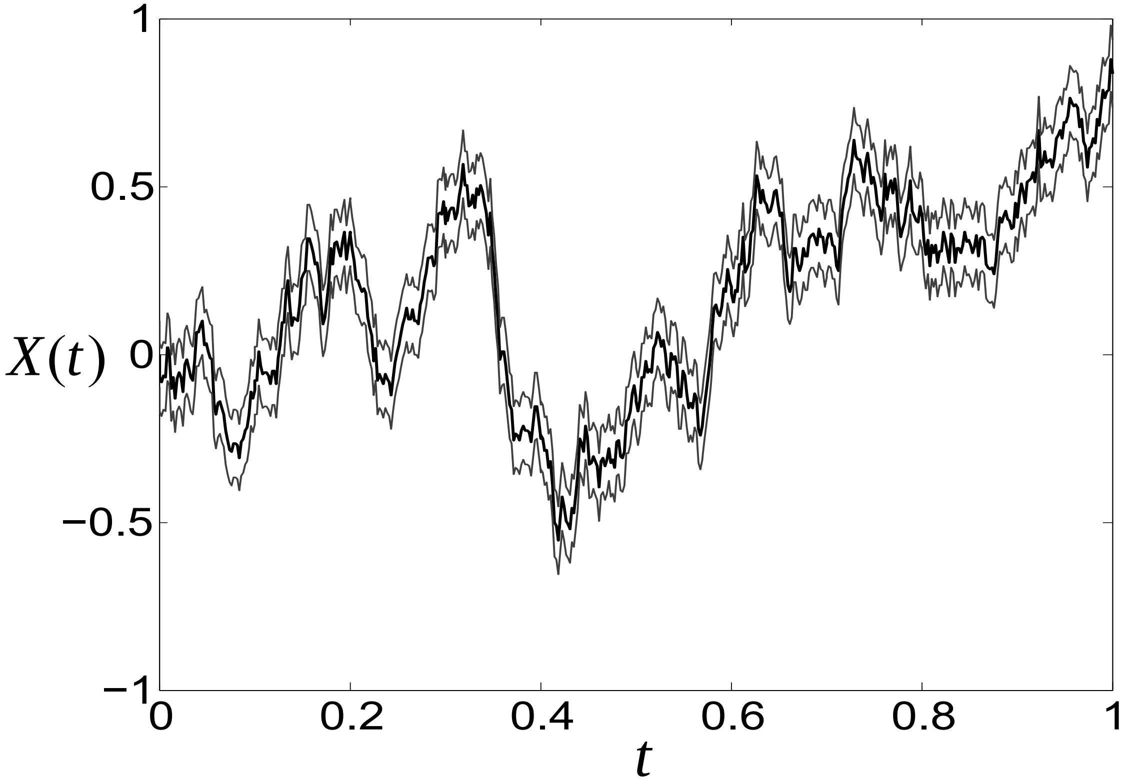


ε-Strong Simulation for SDEs

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t)$$

For any sequence of $0 < \varepsilon_m < \varepsilon_{m-1} < \ldots < \varepsilon_1 < 1$, we can simulate X_{ε_m} conditional on $X_{\varepsilon_1}, \ldots, X_{\varepsilon_{m-1}}$

Tolerance-Enforced Simulation



Sample Path Simulation for SDEs

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t)$$

	One Dimensional	**Multi Dimensional**
Finite collection of time point (Exact)	Beskos and Roberts (2004) Beskos, Papaspiliopoulos and Roberts (2006) Chen and Huang (2013) Giesecke and Smelov (2013) …	Still an open problem…
Sample path (ε-strong)	Beskos, Peluchetti and Roberts (2012)	**Blanchet, Chen and Dong (2014): Today's talk!**

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Sample Path Simulation for SDEs

Beskos and Roberts (2004)

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t)$$

Lamperti Transformation: $Y(t) = \int_y^{X(t)} \frac{1}{\sigma(x)} dx$

↓

$$dY(t) = a(X(t))dt + dB(t)$$

Girsanov's Transformation:

$$\frac{dP}{d\tilde{P}}(\omega) = \exp(\int_0^T a(Y(s))dY(s) - \frac{1}{2}\int_0^T a^2(Y(s))dY(s))$$

↓

$$B(t)$$

ε-Strong Simulation for SDEs

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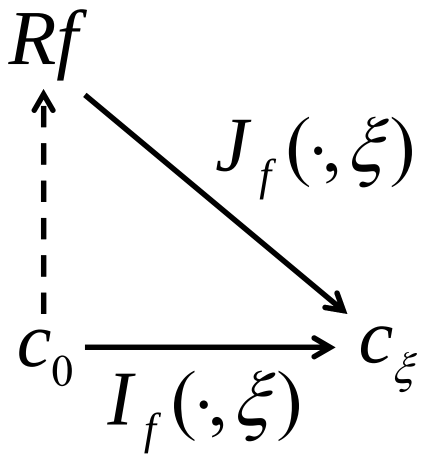
- Theory of Rough Path ➔ construct a continuous mapping



ε-Strong Simulation for SDEs

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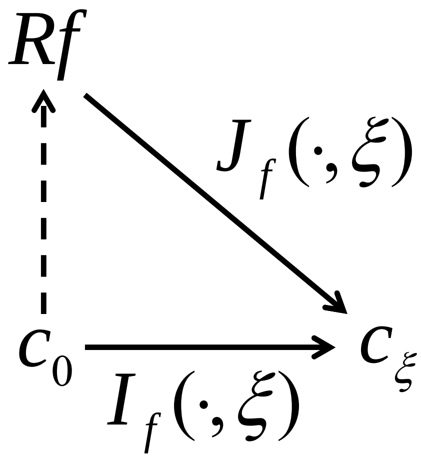
➢ Theory of Rough Path ➔ construct a continuous mapping



➢ Lévy-Ciesielski construction of {B(t): 0≤t≤1}

$$B(t) = W^0\Lambda^0(t) + \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} W_k^n \Lambda_k^n(t) \;\Rightarrow\; B(t) \approx W^0\Lambda^0(t) + \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}} W_k^n \Lambda_k^n(t)$$

Theory of Rough Paths

Lyons (1998)

$$X_i(t) = X_i(0) + \sum_{j=1}^{d'} \int_0^t \sigma_j(X(s))\,dZ_j(t)$$

where Z(t) is continuous but not assume to be differentiable

$$Z(t) \qquad \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} dZ(t_k)\dots dZ(t_1)$$

α-Hölder continuity: $\| Z(t) - Z(s) \|_\infty \le C\,|t-s|^\alpha$

1. α<1/2: $Z(t)$
2. 1/3<α<1/2: $Z(t)$ and $A_{i,j}(s,t) := \int_s^t (Z_i(u) - Z_i(s))\,dZ_j(t)$

Theory of Rough Paths

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α-Hölder continuity: $\| Z(t) - Z(s) \|_\infty \le C \, | t - s |^\alpha$

- $\frac{1}{3} < \alpha \le \frac{1}{2}$: $Z(t)$ and $A_{i,j}(s,t) := \int_s^t (Z_i(u) - Z_i(s))\, dZ_j(t)$

For $0 \le r < s < t \le 1$

a) $A_{i,j}(r,t) = A_{i,j}(r,s) + A_{i,j}(s,t) + (Z_i(s) - Z_i(r))(Z_j(t) - Z_j(s))$

b) $A_{i,j}(s,t) + A_{j,i}(s,t) = (Z_i(t) - Z_i(s))(Z_j(t) - Z_j(s))$

Theory of Rough Paths

$[0,1]$

Dyadic discretization: $D_n = \{\frac{k}{2^n}, k = 0,1,2,...,2^n\}$

$t_k^n = \frac{k}{2^n}$ $\quad$ $t_k^n = t_{2k}^{n+1}$

Mesh of D_n : $\Delta_n = 2^{-n}$

Theory of Rough Paths: Davie's estimates

$$X_i(t) = X_i(0) + \sum_{j=1}^{d'} \int_0^t \sigma_j(X(s))\,dZ_j(t)$$

$$\| Z \|_\alpha = \sup_n \sup_{s,t \in D_n} \frac{\| Z(t) - Z(s) \|_\infty}{|t-s|^\alpha}, \quad \| A \|_{2\alpha} = \sup_n \sup_{s,t \in D_n} \frac{\| A(t) - A(s) \|_\infty}{|t-s|^{2\alpha}}$$

$$\tilde{X}_i^n(t_{k+1}^n) = \tilde{X}_i^n(t_k^n) + \sum_{j=1}^{d'} \sigma(\tilde{X}^n(t_k^n))(Z_j(t_{k+1}^n) - Z_j(t_k^n))$$

$$+ \sum_{j=1}^{d'} \sum_{l=1}^{d} \sum_{m=1}^{d'} \partial_l \sigma_{ij}(\tilde{X}^n(t_k^n)) \sigma_{lm}(\tilde{X}^n(t_k^n)) A_{m,j}(t_k^n, t_{k+1}^n)$$

$$\tilde{X}^n \to X$$

$$\boxed{\| \tilde{X}^n(t) - X(t) \|_\infty \le G(\| Z \|_\alpha, \| A \|_{2\alpha}) \Delta_n^{3\alpha - 1}}$$

Theory of Rough Paths: Davie's estimates

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$$A_{i,j}(t_{k-1}^n, t_k^n) \approx \tilde{A}_{i,j}^n(t_{k-1}^n, t_k^n)$$

$$R_{i,j}^n(t_l^n, t_m^n) = \sum_{k=l+1}^{m} (A_{i,j}^n(t_{k-1}^n, t_k^n) - \tilde{A}_{i,j}^n(t_{k-1}^n, t_k^n))$$

For fixed $1 - \alpha < \beta < 2\alpha$

$$\Gamma_R = \sup_n \sup_{s,t \in D_n} \frac{\| R^n(s,t) \|_\infty}{|t-s|^\beta \, \Delta_n^{2\alpha-\beta}}$$

Theory of Rough Paths: B, Chen, Dong '14

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$$\tilde{X}^n \to X$$

$$\| \tilde{X}^n(t) - X(t) \|_\infty \le G(\| Z \|_\alpha, \| A \|_{2\alpha}, \Gamma_R)\Delta_n^{2\alpha-\beta}$$

Theory of Rough Paths

$$X_i(t) = X_i(0) + \sum_{j=1}^{d'} \int_0^t \sigma_j(X(s))\, dZ_j(t)$$

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Ito: $A_{i,j}(t_{k-1}^n, t_k^n) = \int_{t_{k-1}^n}^{t_k^n} (Z_i(t) - Z_i(t_{k-1}^n))\, dZ_j(t)$

$$\approx (Z_i(t_{k-1}^n) - Z_i(t_{k-1}^n))(Z_j(t_k^n) - Z_j(t_{k-1}^n))$$

$$= 0 = \tilde{A}_{i,j}^n(t_{k-1}^n, t_k^n)$$

$$R_{i,j}^n(t_l^n, t_m^n) = \sum_{k=l+1}^{m} A_{i,j}^n(t_{k-1}^n, t_k^n)$$

For fixed $1 - \alpha < \beta < 2\alpha$,

$$R_{i,j}^n(t_l^n, t_m^n) \le \Gamma_R (m-l)^\beta \Delta_n^{2\alpha}$$

ε-Strong Simulation

Contribution: Simulate upper bounds for

$$\|Z\|_{\alpha} = \sup_n \sup_{s,t\in D_n} \frac{\|Z(t)-Z(s)\|_{\infty}}{|t-s|^{\alpha}}$$

$$\|A\|_{2\alpha} = \sup_n \sup_{s,t\in D_n} \frac{\|A(t)-A(s)\|_{\infty}}{|t-s|^{2\alpha}}$$

$$\Gamma_R = \sup_n \sup_{0\le s<t\le 1} \frac{\|R^n(s,t)\|_{\infty}}{|t-s|^{\beta}\,\Delta_n^{2\alpha-\beta}}$$

$$\Bigg\} \; G(\|Z\|_{\alpha}, \|A\|_{2\alpha}, \Gamma_R)\Delta_N^{2\alpha-\beta} < \varepsilon$$

Jointly with Z(t) for t in the dyadic set D_n for any n

ε-Strong Simulation

Contribution: Simulate upper bounds for

$$\| A \|_{2\alpha} = \sup_n \sup_{s,t \in D_n} \frac{\| A(t) - A(s) \|_\infty}{| t - s |^{2\alpha}}$$

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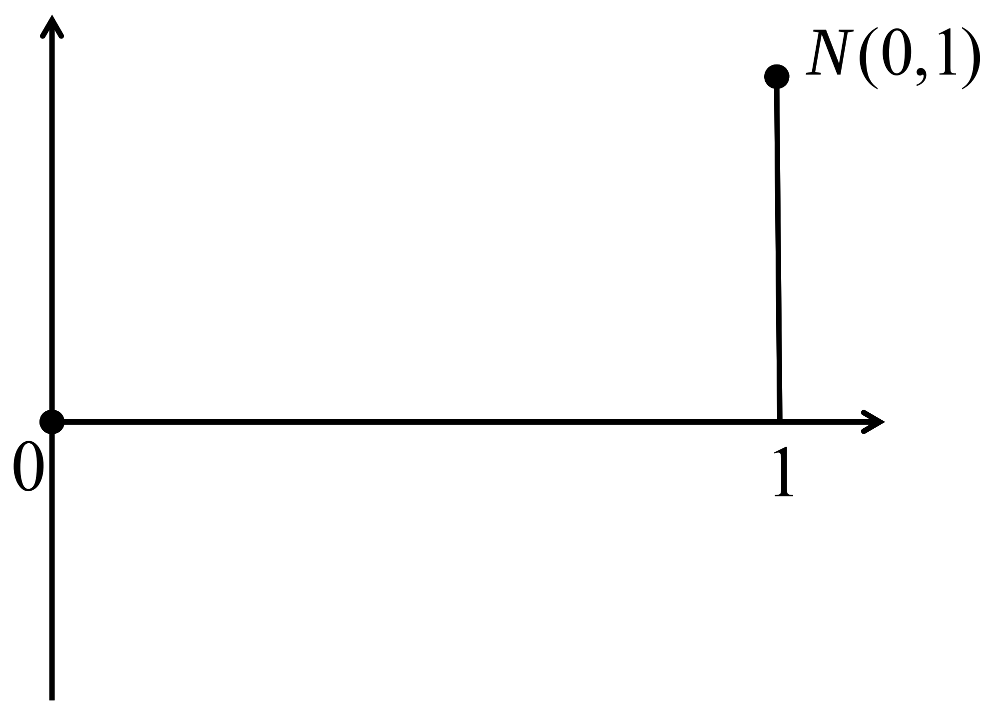
Jointly with Z(t) for t in the dyadic set D_n for any n

Wavelet Construction of Brownian Motion

Lévy-Ciesielski Construction:

$$Z(t) = W^0 \Lambda^0(t) + \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} W_k^n \Lambda_k^n(t)$$



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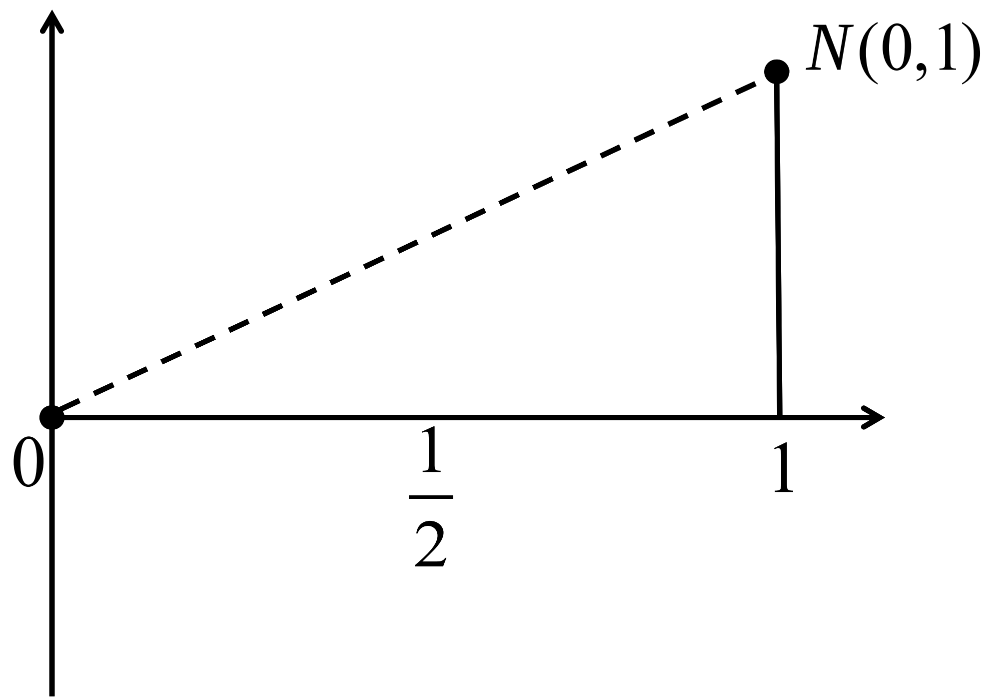

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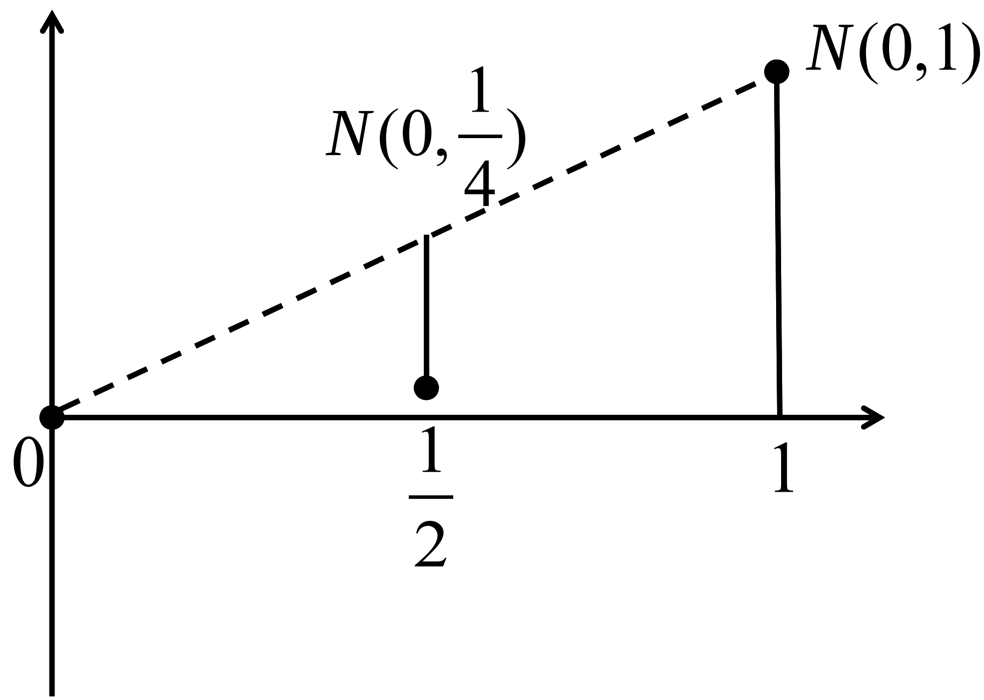

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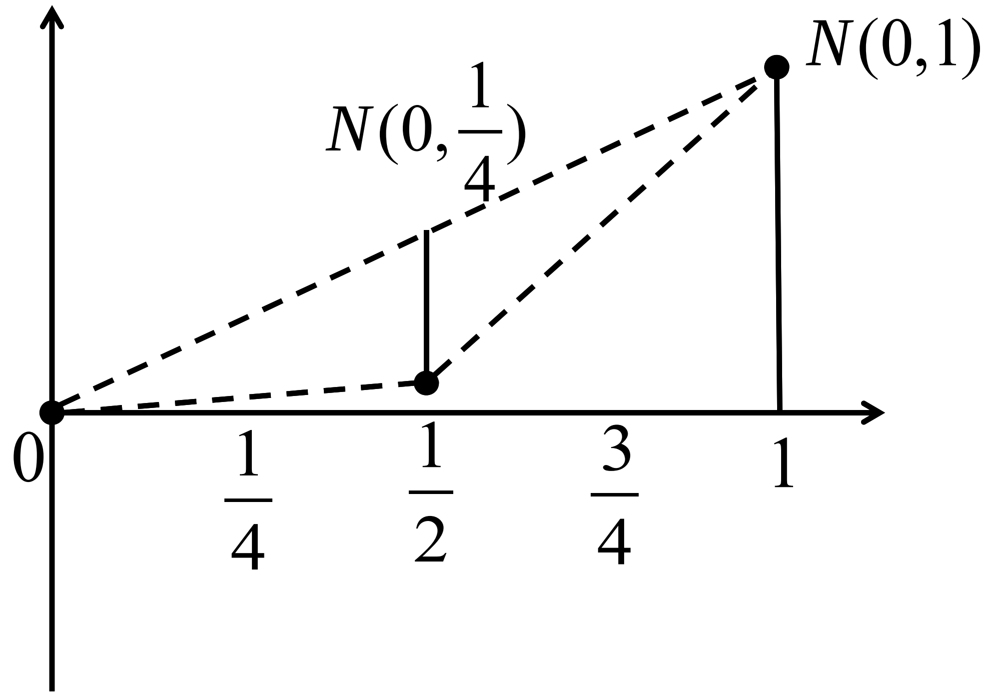

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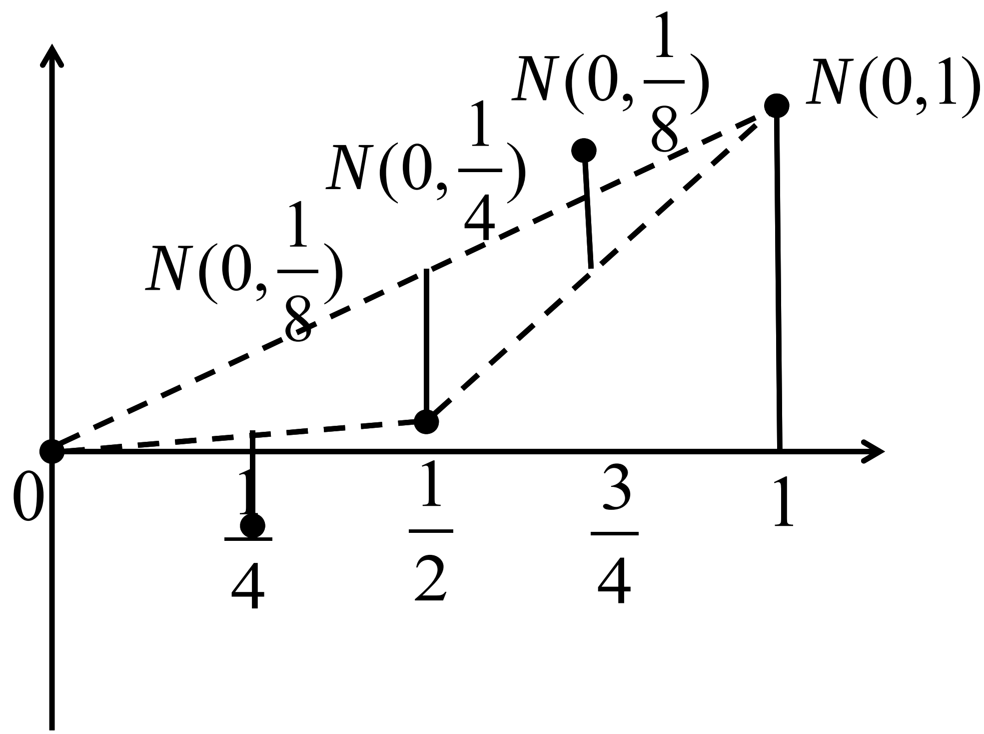

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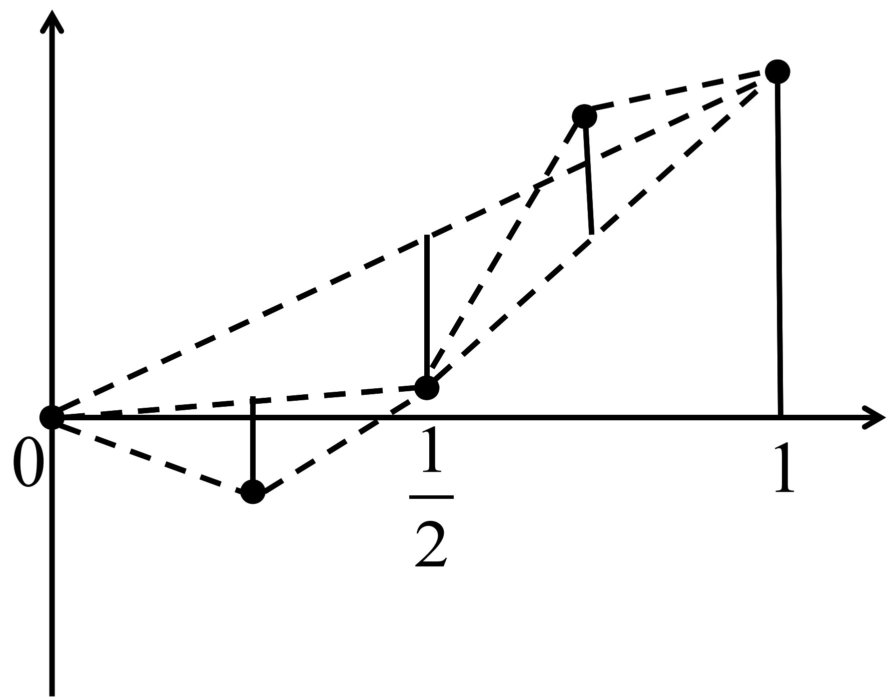

$N(0, 2^{-(n+1)})$

Wavelet Construction of Brownian Motion

Lévy-Ciesielski Construction:

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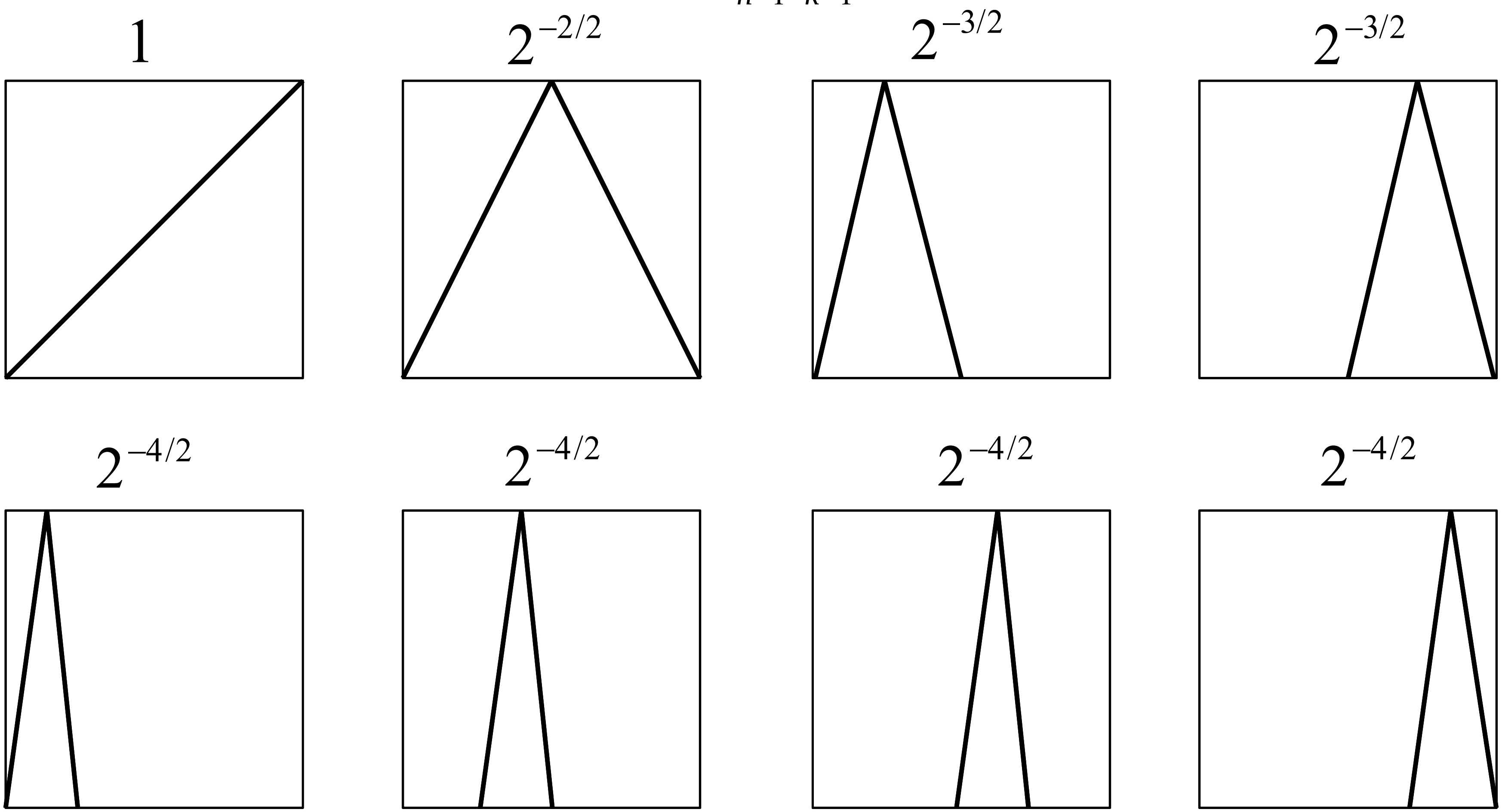


The Idea of Record Breakers for Brownian Path

Record breakers: $\{(n,k,i) : |W_{i,k}^{n}| > 4\sqrt{\log(2^{n-1}+k)}\}$

$$N_1 = \max\{n \geq 1 : 4\sqrt{\log(2^{n-1}+k)} \text{ for some } 1 \leq k \leq 2^{n-1}\}$$

- $EN_1 < \infty$
- $\| Z \|_{\alpha} \leq \sum_{n=1}^{\lceil \log_2 N_1 \rceil} 2^{-n(1/2-\alpha)} \max_k \{W_k^n\} + \dfrac{(\lceil \log_2 N_1 \rceil + 1)^{-1/2(1/2-\alpha)}}{1-2^{-1/2(1/2-\alpha)}}$

Simulation of the Brownian Path

Let $l = 2^{n-1} + k$ and $\mathrm{V}_{i,l} = W_{i,k}^{n}$ for $n \geq 1$

$V_{i,1} = W_i^0$

$\{(i,l) : |V_{i,l}| > a\sqrt{\log l}\}$

$$J_i(0) = 1$$

$$J_i(k) = \{l > J_i(k-1) : V_{i,l} > a\sqrt{\log l}\}$$

$$\gamma_i = \min\{k : J_i(k) = \infty\}$$

Simulation of the Brownian Path

$$J_i(0) = 1$$

$$J_i(k) = \{l > J_i(k-1) : V_{i,l} > a\sqrt{\log l}\}$$

$$\gamma_i = \min\{k : J_i(k) = \infty\}$$

$$U(\infty) = P(J_i(1) = \infty) = \prod_{l=2}^{\infty} P(|V_{i,l}| \leq a\sqrt{\log(l)})$$

$$U(h) = P(J_i(1) > h) = \prod_{l=2}^{h} P(|V_{i,l}| \leq a\sqrt{\log(l)})$$

Simulation of the Brownian Path

$$J_i(0) = 1$$

$$J_i(k) = \{l > J_i(k-1) : V_{i,l} > a\sqrt{\log l}\}$$

$$\gamma_i = \min\{k : J_i(k) = \infty\}$$

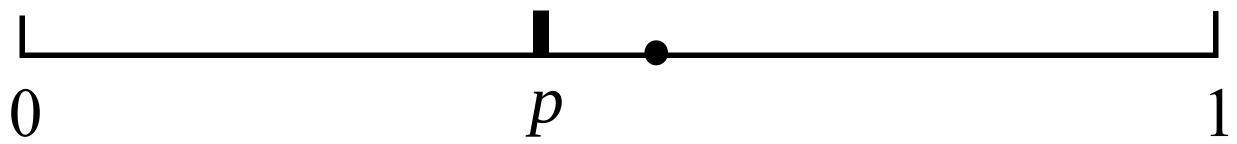
$$p = P(J_i(1) = \infty) = \prod_{l=2}^{\infty} P(|V_{i,l}| \le a\sqrt{\log(l)})$$

$$U(h) = P(J_i(1) > h) = \prod_{l=2}^{h} P(|V_{i,l}| \le a\sqrt{\log(l)})$$

$$P(J_i(1) = \infty) \ge U(h)(1 - h^{1-a^2/2}) = D(h)$$

$$P(J_i(1) = h) = U(h-1) - U(h)$$

Simulation of the Brownian Path



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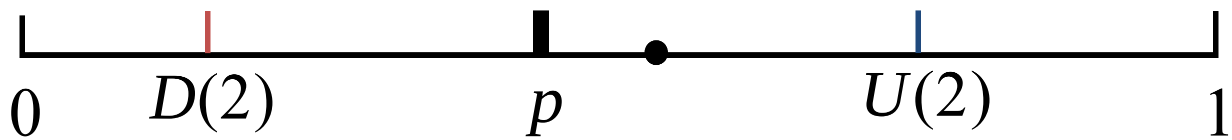

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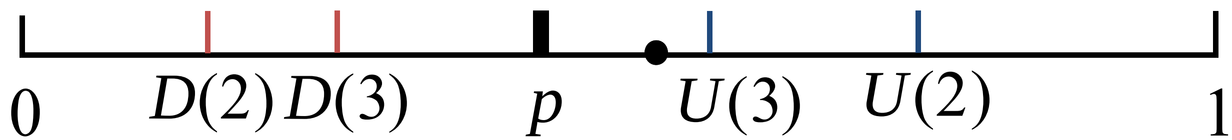

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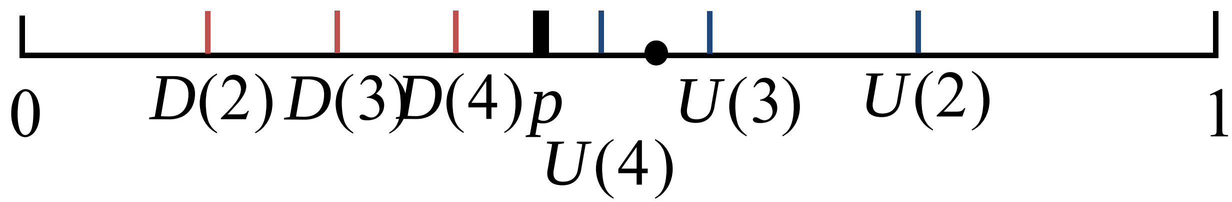

$J(1) = 4$

Simulation of the Brownian Path

1. Find all the record breakers $\{(n,k,i): |W_{i,k}^{n}| > 4\sqrt{\log(2^{n-1}+k)}\}$

2. Simulate $W_{i,k}^{n}$ conditional on the information of the record breakers

Infinite Sum Representation of Lévy Area

$$L_{i,j}^{n}(k) = L_{i,j}^{n}(k-1) + \Xi_i^n(t_{2k-1}^n)\Xi_j^n(t_{2k}^n)$$

Let $\Xi_i^n(t_r^n) = Z_i(t_r^n) - Z_i(t_{r-1}^n)$

$$\Xi_i^n(t_{2k-1}^n) = \frac{1}{2}\Xi_i^{n-1}(t_k^{n-1}) + \Delta_{n+1}^{1/2} W_{i,k}^n$$

$$\Xi_i^n(t_{2k}^n) = \frac{1}{2}\Xi_i^{n-1}(t_k^{n-1}) - \Delta_{n+1}^{1/2} W_{i,k}^n$$

$$A_{i,j}(t_k^n, t_{k+1}^n) = \sum_{h=n+1}^{\infty} \left(L_{i,j}^h(2^{h-n}(k+1)) - L_{i,j}^h(2^{h-n}k)\right)$$

$$R_{i,j}(t_k^n, t_{k'}^n) = \sum_{h=n+1}^{\infty} \left(L_{i,j}^h(2^{h-n}k') - L_{i,j}^h(2^{h-n}k)\right)$$

The Idea of Record Breakers for Lévy Area

Record breakers: $\{(n,k,k',i,j):|L_{i,j}^n(k')-L_{i,j}^n(k)|>(k'-k)^\beta \Delta_n^{2\alpha}\}$

$N_2 = \max\{n \geq 1 : |L_{i,j}^n(k')-L_{i,j}^n(k)|>(k'-k)^\beta \Delta_n^{2\alpha} \text{ for some } 0 \leq k < k' \leq 2^{n-1}\}$

- $P(N_2 < \infty) = 1$
- $\Gamma_R \leq \dfrac{2^{-(2\alpha-\beta)}}{1-2^{-(2\alpha-\beta)}} \max_{n \leq N_2} \max_{0 \leq k < k' \leq 2^{n-1}} \dfrac{\| L^n(k') - L^n(k) \|_\infty}{(k'-k)^\beta \Delta_n^{2\alpha}}$

The Idea of Record Breakers for Lévy Area

Record breakers: $\{(n,k,k',i,j):|L^n_{i,j}(k')-L^n_{i,j}(k)|>(k'-k)^{\beta}\Delta_n^{2\alpha}\}$

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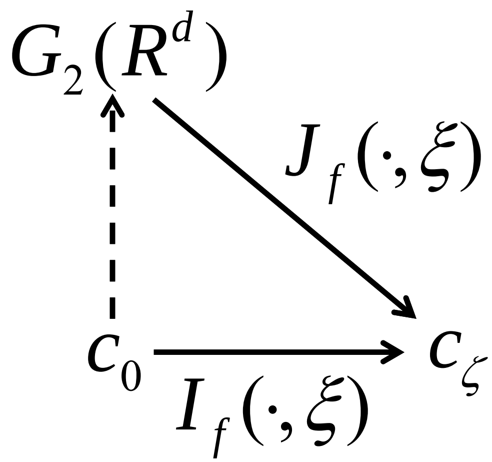
- $P(N_2<\infty)=1$
- $\Gamma_R\le\dfrac{2^{-(2\alpha-\beta)}}{1-2^{-(2\alpha-\beta)}}\max_{n\le N_2}\max_{0\le k<k'\le 2^{n-1}}\dfrac{\|L^n(k')-L^n(k)\|_\infty}{(k'-k)^{\beta}\Delta_n^{2\alpha}}$
- $\|A\|_{2\alpha}\le\Gamma_R\dfrac{2}{1-2^{-2\alpha}}+\|Z\|_{\alpha}^2\dfrac{2^{1-\alpha}}{1-2^{-\alpha}}$

Summary

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dZ(t)$$

- Theory of Rough Path ➔ construct a continuous mapping



$$Z(t)$$

$$A_{i,j}(s,t) = \int_s^t (Z_i(u) - Z_i(s))dZ_j(u)$$

- Lévy-Ciesielski construction of {Z(t): 0≤t≤1}

$$Z(t) \approx W^0\Lambda^0(t) + \sum_{n=1}^{N}\sum_{k=1}^{2^{n-1}} W_k^n \Lambda_k^n(t)$$

Thank you !

Record Breakers

Negative drifted random walk

$$S_n = X_1 + X_2 + \ldots + X_n \text{ with } E[X_1] < 0$$

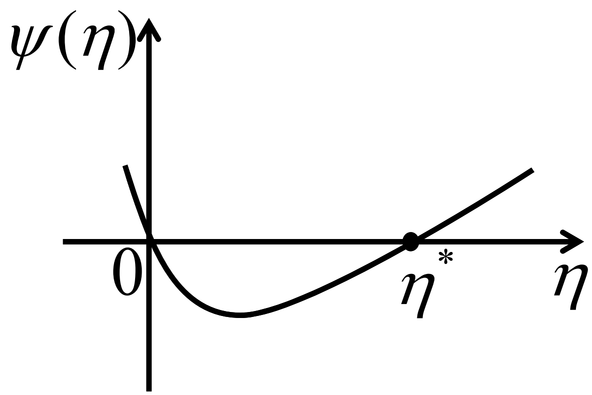
$$T_a = \inf\{n \geq 0 : S_n \geq a\}$$

$$P(T_a = \infty) > 0$$

Negative drifted random walk

$$f_\eta(y) = \exp(\eta y - \psi(\eta)) f(y) \text{ where } \psi(\eta) := \log E\left[\exp(\eta Y_i)\right]$$

$$\eta^* : \psi(\eta^*) := \log E\left[\exp(\eta^* Y_i)\right] = 0$$



$$f_{\eta^*}(y) = f(y)\exp(\eta^* y)$$

$$P(T_a < \infty) = E_{\eta^*}\left[\exp(-\eta^* S_{T_a}) 1\{T_a < \infty\}\right] = E_{\eta^*}\left[\exp(-\eta^* S_{T_a})\right]$$

$$U \leq \exp(-\eta^* S_{T_a})$$

Simulation of the Lévy Area

$$P_n[\bullet] = P[\bullet \mid \mathbb{F}_n]$$

$$L_{i,j}^{n+m}(k) = \sum_{r=1}^{k} \Xi_i^{n+m}(t_{2r-1}^{n+m})\Xi_j^{n}(t_{2r}^{n+m})$$

$$\Xi_i^n(t_{2k-1}^n) = \frac{1}{2}\Xi_i^{n-1}(t_k^{n-1}) + \Delta_{n+1}^{1/2} W_{i,k}^n$$

$$\Xi_i^n(t_{2k}^n) = \frac{1}{2}\Xi_i^{n-1}(t_k^{n-1}) - \Delta_{n+1}^{1/2} W_{i,k}^n$$

$$E_n[E_{n+1}[\ldots E_{n+m-1}[\exp(\theta_0\{L_{i,j}^{n+m}(k') - L_{i,j}^{n+m}(k)\})]\ldots]]$$

Simulation of the Lévy Area

1. Simulate $M = m, I = i, J = j, K = k, K' = k'$
2. Simulate $\omega_{n:n+m}$
3. Simulate U □ Uniform[0,1]

   If $U \leq L_n(m, i, j, k, k', \omega_{n:n+m})$ output $\omega_{n:n+m}$

   else we got $\mathrm{N}_2 = n$