Optimal Transport in Risk Analysis

Jose Blanchet (based on work with Y. Kang and K. Murthy)

Stanford University (Management Science and Engineering), and Columbia University (Department of Statistics and Department of IEOR).

Goal:

Goal: Present a comprehensive framework for decision making under model uncertainty...

This presentation is an invitation to read these two papers:

https://arxiv.org/abs/1604.01446

https://arxiv.org/abs/1610.05627

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- B is a set which models bankruptcy.
- **Problem:** Model (P_{true}) may be complex, intractable or simply unknown...

A Distributionally Robust Risk Analysis Formulation

- **Our solution:** Estimate u_T by solving

$$\sup_{D_c(P_0,P)\leq\delta} P\left(R\left(t\right) \in B \text{ for some } t \in \left[0, T\right]\right),$$

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- δ is the distributional uncertainty size.
- $D_c\left(\cdot\right)$ is the distributional uncertainty region.

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- *Want to preserve advantages of using* P_0*.*
- Want a way to estimate δ.

Connections to Distributionally Robust Optimization

- Standard choices based on divergence (such as Kullback-Leibler) - Hansen & Sargent (2016)

$$D\left(v||\mu\right) = E_v\left(\log\left(\frac{dv}{d\mu}\right)\right).$$

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- **We advocate using optimal transport costs (e.g. Wasserstein distance).**

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- $\mu(\cdot)$ and $v(\cdot)$ Borel probability measures defined on $\mathcal{S}_X$ and $\mathcal{S}_Y$.
- Given π a Borel prob. measure on $\mathcal{S}_X \times \mathcal{S}_Y$,

$$\pi_X(A) = \pi(A \times \mathcal{S}_Y) \quad \text{and} \quad \pi_Y(C) = \pi(\mathcal{S}_X \times C).$$

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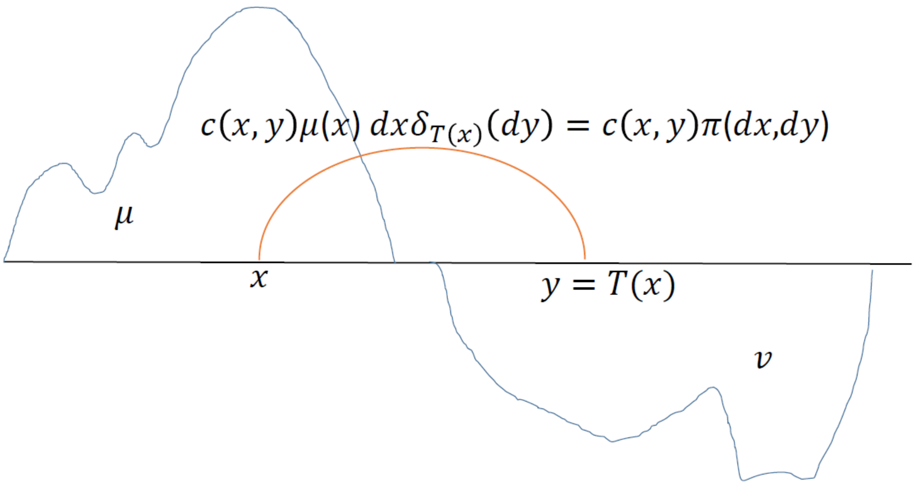
- *Kantorovich's problem is a "nice" infinite dimensional linear programming problem.*

Illustration of Optimal Transport Costs



Fundamental Theorem of Robust Performance Analysis

Theorem (B. and Murthy (2016))

Suppose that $c(\cdot)$ is lower semicontinuous and that $h(\cdot)$ is upper semicontinuous with $E_{P_0}|f(X)| < \infty$. Then,

$$\sup_{D_c(P_0,P)\leq\delta} E_P(f(Y)) = \inf_{\lambda\geq 0} E_{P_0}[\lambda\delta + \sup_z\{f(z) - \lambda c(X,z)\}].$$

Moreover, (π_*) and dual λ_* are primal-dual solutions if and only if

$$\begin{aligned} f(y) - \lambda_* c(x,y) &= \sup_z\{f(z) - \lambda_* c(x,z)\} \quad (x,y)\text{-}\pi_* \text{ a.s.} \\ \lambda_*(E_{\pi_*}[c(X,Y) - \delta]) &= 0. \end{aligned}$$

Implications for Risk Analysis

Theorem (B. and Murthy (2016))

Suppose that $c(\cdot)$ *is lower semicontinuous and that* B *is a closed set. Let* $c_B(x) = \inf\{c(x,y) : y \in B\}$, *then*

$$\sup_{D_c(P_0,P)\leq\delta} P(Y \in B) = P_0\left(c_B(X) \leq 1/\lambda^*\right),$$

where $\lambda^* \geq 0$ *satisfies (under mild assumptions on* $c_B(X)$*)*

$$\delta = E_0\left[c_B(X)\, I\left(c_B(X) \leq 1/\lambda^*\right)\right].$$

- Suppose that

$$
\begin{aligned}
c(x,y) &= d_J(x(\cdot), y(\cdot)) = \text{Skorokhod } J_1 \text{ metric.} \\
&= \inf_{\phi(\cdot) \text{ bijection}} \left\{ \sup_{t\in[0,1]} |x(t) - y(\phi(t))|, \sup_{t\in[0,1]} |\phi(t) - t| \right\}.
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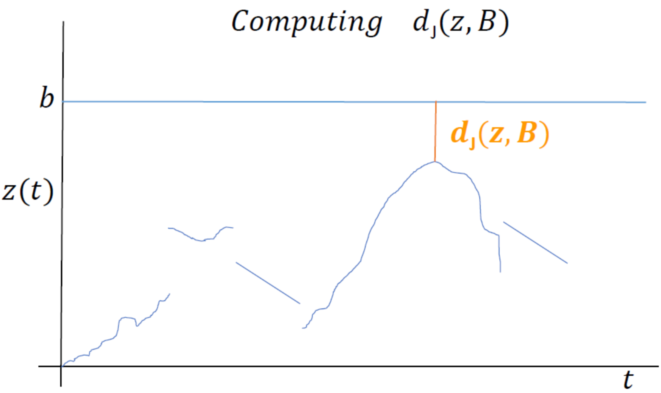
- **Let $P_0(\cdot)$ be the Wiener measure want to compute**

$$
\sup_{D_c(P_0,P)\leq\delta} P(Z \in B_b).
$$

Application 1: Computing Distance to Bankruptcy



- **So:** $\{c_{B_b}(z) \leq 1/\lambda_*\} = \{\sup_{t\in[0,1]} z(t) \geq b - 1/\lambda^*\}$, and

$$\sup_{D_c(P_0,P)\leq\delta} P(Z \in B_b) = P_0\left(\sup_{t\in[0,1]} Z(t) \geq b - 1/\lambda^*\right).$$

- Note **any coupling** π so that $\pi_X = P_0$ and $\pi_Y = P$ satisfies

$$D_c\left(P_0, P\right) \leq E_\pi\left[c\left(X, Y\right)\right] \approx \delta.$$

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- So use any coupling between *evidence* and P_0 or expert knowledge.
- We discuss choosing δ non-parametrically in a moment

- Given arrivals and claim sizes let $Z(t) = m_2^{-1/2} \sum_{k=1}^{N(t)} (X_k - m_1)$

Algorithm 1 To embed the process $(Z(t) : t \geq 0)$ in Brownian motion $(B(t) : t \geq 0)$
Given: Brownian motion $B(t)$, moment m_1 and independent realizations of claim sizes $X_1, X_2, \ldots$

Initialize $\tau_0 := 0$ and $\Psi_0 := 0$. For $j \geq 1$, recursively define,

$$\tau_{j+1} := \inf\left\{ s \geq \tau_j : \sup_{\tau_j \leq r \leq s} B_r - B_s = X_{j+1} \right\}, \text{ and } \Psi_j := \Psi_{j-1} + X_j.$$

Define the auxiliary processes

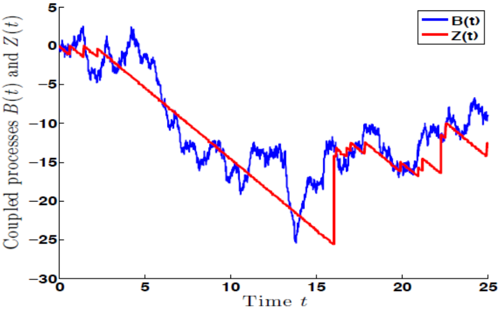
$$\tilde{S}(t) := \sum_{j>0} \sup_{\tau_j \leq s \leq t} B(s) \mathbf{1}(\tau_j \leq t < \tau_{j+1}) \text{ and } \tilde{N}(t) := \sum_{j \geq 0} \Psi_j \mathbf{1}(\tau_j \leq t < \tau_{j+1}).$$

Let $A(t) := \tilde{N}(t) + \tilde{S}(t)$, and identify the time change $\sigma(t) := \inf\{s : A(s) = m_1 t\}$. Next, take the time changed version $Z(t) := \tilde{S}(\sigma(t))$.

Replace $Z(t)$ by $-Z(t)$ and $B(t)$ by $-B(t)$.

Application 1: Coupling in Action

Figure 4. A coupled path output by Algorithm 1

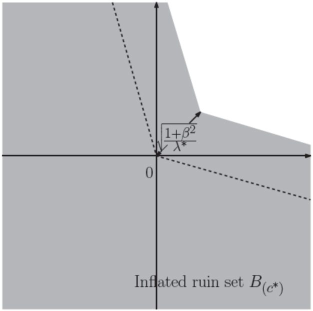


Application 1: Numerical Example

- Assume Poisson arrivals.
- *Pareto claim sizes with index* **2.2** $(P(V > t) = 1/(1+t)^{2.2})$.
- Cost $c(x, y) = d_J(x, y)^2$ <– note power of 2.
- Used Algorithm 1 to calibrate (estimating means and variances from data).

b	$\frac{P_0(\text{Ruin})}{P_{true}(\text{Ruin})}$	$\frac{P^*_{robust}(\text{Ruin})}{P_{true}(\text{Ruin})}$
100	1.07×10^{-1}	12.28
150	2.52×10^{-4}	10.65
200	5.35×10^{-8}	10.80
250	1.15×10^{-12}	10.98

- Paper: Quantifying Distributional Model Risk via Optimal Transport (B. & Murthy '16) **https://arxiv.org/abs/1604.01446** contains more applications
- Multidimensional risk processes (explicit evaluation of $c_B(x)$ for d_J metric).
- Control: $\min_\theta \sup_{P:D(P,P_0)\leq\delta} E[L(\theta, Z)]$ <– robust optimal reinsurance.



(b)Computation of worst-case ruin using the

Connection to machine learning helps further understand why optimal transport costs are sensible choices...

Paper:
Robust Wasserstein Profile Inference and Applications to Machine Learning (B., Murthy & Kang '16)
https://arxiv.org/abs/1610.05627

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$$MSE(\beta) = \min_{\beta} \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \beta^T X_i \right)^2.$$

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- Apply the distributionally robust estimator based on optimal transport.

Connection to Sqrt-Lasso

Theorem (B., Kang, Murthy (2016)) Suppose that

$$c\left(\left(x,y\right),\left(x',y'\right)\right)=\begin{cases}\left\|x-x'\right\|_q^2 & \text{if} \quad y=y' \\ \infty & \text{if} \quad y\neq y'\end{cases}.$$

Then, if $1/p+1/q=1$

$$\max_{P:D_c(P,P_n)\leq\delta} E_P^{1/2}\left(\left(Y-\beta^T X\right)^2\right)=\sqrt{MSE\left(\beta\right)}+\sqrt{\delta}\left\|\beta\right\|_p^2.$$

Remark 1: This is sqrt-Lasso (Belloni et al. (2011)).
Remark 2: Also representations for support vector machines, LAD lasso, group lasso, adaptive lasso, and more!

For Instance... Regularized Estimators

Theorem (B., Kang, Murthy (2016)) Suppose that

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Then,

$$\sup_{P:\ \mathcal{D}_c(P,P_n)\leq\delta} E_P\left[\log(1+e^{-Y\beta^T X})\right] = \frac{1}{n}\sum_{i=1}^{n}\log(1+e^{-Y_i\beta^T X_i}) + \delta\left\|\beta\right\|_p.$$

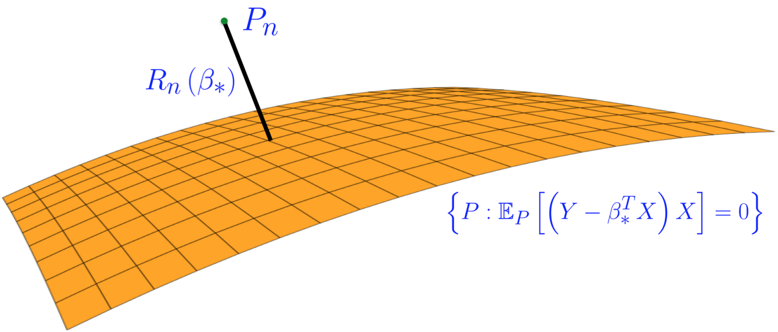
Remark 1: This is regularized logistic regression (see also Esfahani and Kuhn 2015).

Connections to Empirical Likelihood

Paper:
https://arxiv.org/abs/1610.05627

Also chooses δ *optimally* introducing
an extension of Empirical Likelihood
called "Robust Wasserstein Profile Inference".

The Robust Wasserstein Profile Function



Pick $\delta = 95\%$ quantile of $R_n(\beta_*)$ and we show that

$$nR_n(\beta_*) \approx_d \frac{E[e^2]}{E[e^2] - (E|e|)^2} \|N(0, Cov(X))\|_q^2.$$

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- Extensions of Empirical Likelihood & connections to optimal regularization.